CERN-TH.7174/94
BONN-IR-94-02
SLAC–PUB–hep-th/9404157
February, 1994

STRINGS, BLACK HOLES AND CONFORMAL FIELD THEORY*

Katrin Becker

Theory Division, CERN
CH-1211, Geneva 23, Switzerland

and

Physikalisches Institut, Universität Bonn,
Nussallee 12, D-53115 Bonn, Germany.

ABSTRACT

The SL(2,ℝ)/U(1) gauged Wess-Zumino-Witten model is an exact conformal field theory describing a black hole in two-dimensional space-time. The free field approach of Bershadsky and Kutasov is a suitable formulation of this CFT in order to compute physically interesting quantities of this black hole. We find the space-time interpretation of this model for $k = 9/4$ and show that it reproduces the metric and the dilaton found by Dijkgraaf, E. Verlinde and H. Verlinde in the mini-superspace approximation. We compute the two- and three-point functions of tachyons interacting in the black hole background and analyse in detail the form of the four-point tachyon scattering amplitude. We discuss the connection to the $c = 1$ matrix model and the deformed matrix model of Jevicki and Yoneya.

⋆ PhD Thesis at the University of Bonn.

Dedicated to my mother and
to the memory of my father

STRINGS, BLACK HOLES AND CONFORMAL FIELD THEORY

Katrin Becker

Theory Division, CERN

CH-1211, Geneva 23, Switzerland

and

Physikalisches Institut, Universität Bonn,

Nussallee 12, D-53115 Bonn, Germany.

Contents

1. INTRODUCTION

1.1 String Theory and Black Holes

The classical theory of general relativity predicts the existence of black holes as solutions to Einstein's equations. These are regions of space-time from which, classically, it is not possible to escape to infinity; they are separated from the exterior by a null hypersurface called the event horizon. General relativity provides an adequate description of black holes that are much bigger than the Planck mass. However, much smaller black holes could have been formed in the early universe. For these black holes a description in terms of general relativity breaks down and it has to be replaced by a quantum theory of gravity.

It is undoubtedly true that such a formulation in terms of quantum gravity is important to solve many puzzles connected with the "information loss" that takes place in the presence of a black hole. The classical "no hair" theorem states that a black hole can be characterized by just three conserved charges that are: mass, angular momentum and electric charge. In principle all the information on the matter that formed the black hole is therefore hidden behind the horizon. This is not a worry in the purely classical theory, since the information is still present behind the horizon, even if we cannot get at it. The situation is different in the quantum theory. In 1974 Hawking [1] discovered that due to quantum mechanical pair production at the horizon a black hole can radiate and loose mass. The outgoing radiation is thermal, as if the black hole were a black body with a temperature proportional to the surface gravity. Eventually the black hole evaporates completely and carries all the information of the collapsing matter with it. This implies that pure states can evolve into mixed states and the laws of quantum mechanics would be violated. Again, we expect that quantum gravity will help us to solve this paradox. When the black hole is of the order of the Planck mass we expect corrections from quantum gravitational effects, so that Hawking's semiclassical calculation is no longer valid. There are several possibilities for what could happen next in the evaporation process [2]:

- ▹ Hawking's proposal [3] is radical and it states that the black hole evaporates completely and carries all the information on the in-falling matter with it. Quantum mechanics is therefore not deterministic and our basic physical laws have to be reformulated.
- ▹ The black hole does not evaporate completely but leaves a stable remnant that could contain the information. The final state (remnant plus radiation) is pure. This proposal raises some conceptual difficulties because, for example, it clearly violates CPT since the black hole can form but never disappear completely.
- ▹ The black hole disappears completely but the outgoing radiation is correlated with the in-falling matter and radiation in such a way that the final state, consisting of pure radiation, is pure. Quantum coherence could be restored by the radiation emitted in the final stages of the evaporation where unknown laws of quantum gravity become relevant.
- ▹ None of the above.

Quantum gravity plays an important near the singularity of the black hole. As long as the singularity is hidden inside the event horizon it does not affect the exterior world, because the two regions are causally disconnected. The situation is different if naked singularities do appear since in this case a description in terms of general relativity is no longer valid. This is clearly unsatisfactory and several solutions to this problem have been proposed. One of them is the Cosmic Censorship Hypothesis of Penrose, which states that naked singularities do not appear in nature. However, although we have no counter-examples to this conjecture a general proof is still lacking, so that we have to look for a different way to solve this breakdown of general relativity. One of the answers could be that these singularities would not be present, if instead of considering a classical theory of gravitation, a description in terms of quantum gravity were made.

From this discussion it becomes clear that quantum gravity plays an essential role in every theory of extremely strong gravitational fields such as a black hole.

This brings us to the question of whether we know how to reconcile general relativity with quantum mechanics. As far as four-dimensional gravity is concerned, we encounter many difficulties when we try to quantize the theory. All of them are related to the fact that quantum gravity is a non-renormalizable theory in four dimensions. At present our most promising candidate to be a consistent theory of quantum gravity is string theory. According to this it appears natural to analyse the connection between string theory and black hole physics.

In this context two interesting 2D black hole toy models have been extensively studied during the last three years. Here the technical complexities of four-dimensional quantum black holes are simplified drastically, while the conceptual difficulties of the problem are still kept. One of these models is the $(1+1)$-dimensional dilaton gravity model of Callan, Giddings, Harvey and Strominger [4], in which the gravitational collapse can be studied using a two-dimensional quantum field theory. This model incorporates Hawking radiation, and it solves the β-function equations of the string to first order in the string coupling constant. It has been successfully solved in the semiclassical limit while at the quantum level many puzzles are still unresolved.

Analysing the classical solutions of string theory [5,6] it has been found that the graviton-dilaton equations admit a Schwarzschild-like solution and this was the starting point to find the exact solution of the β-function equations. If we are considering the gravitational applications of string theory it is important to be able to go beyond the leading orders in the expansion in α' (the string coupling constant). The leading order may not correctly describe the strong curvature regions near singularities. Witten [5] proposed that the exact conformal field theory (to all orders in α') that describes the above black hole solution can be formulated in terms of an SL(2,$\mathbb{R}$) gauged Wess-Zumino-Witten (WZW) model. Depending on whether the subgroup that is gauged is compact or not we get the Euclidean version of the black hole or its Lorentzian continuation. In the semiclassical limit $k \to \infty$ (where k is the level of the Kac-Moody algebra) the Minkowski version of this model has a maximally extended space-time analogue to the Schwarzschild black hole of four-dimensional general relativity.

The exact background metric of Witten's black hole has been determined by Dijkgraaf, E. Verlinde and H. Verlinde [7]. This has been done in the "L_0-approach" in which a mini-superspace description of the problem was made. We are going to explain this quantum mechanical approach in some detail in section 3.4. It has been checked by Tseytlin [8] and by Jack et al. [9] that this metric indeed solves the β-function equations of the string perturbatively up to three and four loops. The maximally extended space-time of this geometry has been considered by Perry and Teo [10] and by Yi [11]. It consists of an infinite number of universes connected by wormholes. There are no singularities.

It has been claimed by Witten [5,12] that the black hole looses mass due to Hawking radiation. The end-point of this radiation process is described by the standard $c = 1$ matrix model that can be regarded as an analogue of the extreme Reissner-Nordström black hole. However, it has been argued by Seiberg and Shenker [13] that the black hole mass operator has a non-normalizable wave function, implying the stability of the black hole. The precise relation between the black hole conformal field theory and the standard non-critical string theory is an open problem.

From the matrix model point of view there have been two different main approaches in the literature to describe the black hole. The first one uses the formulation of the $c = 1$ conventional matrix model [14,15] in terms of non-relativistic fermions. In this approach the black-hole singularity is identified with the Fermi surface in the phase space of the fermion and it is a consequence of the semi-classical approximation. They concluded that stringy quantum effects wash out the classical singularity. In the approach of Jevicki et al. [16] it is conjectured that Witten's black hole is described in terms of a deformation of the usual $c = 1$ matrix model. The working hypothesis of these authors consists of two key ingredients, namely a non-local redefinition of the tachyon field and a deformation of the $c = 1$ matrix model at $\mu = 0$. The space-time interpretation of this model has not been worked out so far, so that the relation to the SL(2,$\mathbb{R}$)/U(1) gauged WZW is unclear. However, the "deformed" matrix model is interesting in its own right because it is a different model than the conventional $c = 1$ model, that can be solved non-perturbatively using a free fermion picture.

The formulation in terms of matrix models is important since these models allow us to take into account higher genus effects [17], even in the (physically more interesting) supersymmetric theories [18,19,20,21].

Given these powerful non-perturbative formulations, it is certainly quite important to understand the precise relation between non-critical string theory and the black hole CFT in the continuum approach, because this will help us to understand better the discrete approach. This is one of the purposes of this thesis.

The connection between the BRST cohomologies of both theories has been studied extensively in the literature. Distler and Nelson [22] used the representation theory of SL(2,$\mathbb{R}$) to show that the black hole has the same physical states as the $c = 1$ theory plus some new discrete states that do not have any counterpart in $c = 1$. However, they also stated that it is possible that the real spectrum of the black hole is actually only a truncation of what is allowed by representation theory so that both cohomologies could in principle agree. The structure of discrete states for the black hole is as rich as the one of $c = 1$. So for example, extending standard techniques of the Kac-Moody current algebras to the non-compact case, Chaudhuri and Lykken [23] constructed the elements of the ground ring and showed that the discrete states of the black hole form a W_∞-type algebra. Eguchi et al. [24] have shown that the free-field BRST cohomologies of $c = 1$ coupled to Liouville theory and the black hole coincide. The important point in their proof is the existence of an isomorphism between the states of $c = 1$ and those of the black hole. It can be derived from the fact that both energy-momentum tensors agree up to BRST commutators. The appearance of the W_∞ algebra and the ground ring is then as natural as it is for $c = 1$. The role that the W_∞ algebra plays in the black hole context has been the subject of many controversies. Ellis et al. [25] have argued that due to the infinite number of conserved quantities quantum coherence would be maintained during the black hole evaporation process. A different point of view has been presented in ref. [26].

In spite of this extensive analysis of the cohomologies, the problems of calculating the $\mathcal{S}$-matrix in the full quantum field theory of tachyons in the black hole background and its relation to the scattering amplitudes of $c = 1$ have not been considered so far. This is one of the problems that we are going to address here.

We will consider Witten's 2D black hole using an explicit representation of the fields in terms of Wakimoto coordinates. This representation of Witten's black hole has been introduced by Bershadsky and Kutasov [27] and it provides us with a suitable prescription of how to evaluate scattering amplitudes using a free-field approach. The relation to the gauged WZW model is very clear in this formulation, specially through the Gauss decomposition [28, 29] and the space-time interpretation for finite k [30], for which we will later show that we get agreement with the metric found be Dijkgraaf et al. [7].

The evaluation of scattering amplitudes can therefore be done with familiar techniques used in the Liouville approach to 2D quantum gravity. After the zero mode integration of the fields [31], the path integral is reduced to the one of a free theory, where the screening charges are just new insertions. The amplitudes that can be calculated in this way are those where the number of screening charges is an integer. More general amplitudes are determined by analytic continuation from the integers. The SL(2,$\mathbb{R}$) screening charge has been identified with the operator that creates the mass of the black hole [24,27]. In ref. [27] it has been shown that the bulk amplitudes of tachyons, i.e. those amplitudes that do not need screening charges to satisfy the charge conservation, agree with tachyon correlation functions of $c = 1$ matter coupled to Liouville theory. This correspondence is easy to understand if one takes into account that the representation of Witten's black hole in terms of Wakimoto coordinates can be formulated in the $c = 1$ language. The tachyon vertex operators have identical form in $c = 1$ and the black hole, the only difference is the perturbation considered in the action. While for ordinary Liouville theory the perturbation is a tachyon operator (the cosmological constant), the black hole mass operator corresponds to the discrete state $W^-_{1,0}\bar{W}^-_{1,0}$ of $c = 1$. This is an operator on the wrong branch whose wave function is not normalizable. However, we will later see that correlation functions of tachyon vertex operators in both theories do indeed have very similar features, also if one takes these two different perturbations into account [32,30]. We will calculate explicitly the two-, three-, and four-point functions, where the remarkable analogy between the scattering amplitudes will become clear. Our methods can be applied to N-point functions.

Finally, we would like to mention that a connection between $c = 1$ and the black hole has been considered by Martinec and Shatashvili [33] from a different point of view. They analysed the Hamiltonian path integral quantization of the gauged WZW model and showed that there appears a relation to the Liouville theory coupled to a free scalar field. This connection could in principle be used to determine the scattering amplitudes of the black hole in terms of the $c = 1$ correlators. It would be nice to see a direct connection between our results and those of ref. [14,15,33].

1.2 Outline of the Thesis

This thesis is organized as follows:

II. In chapter 2 we are going to introduce some basic notions that we need in order to describe the propagation of strings in a black hole background. First of all we are going to see in section 1.1 how black holes appear as a solution of Einstein's equations in the theory of classical general relativity. The Schwarzschild solution of the gravitational field equations in empty space is explained in some detail as well as its generalizations characterized by the mass, charge and angular momentum. In section 1.2 we will see how the Polyakov action that describes string propagation in flat space-time has to be generalized in order to consider strings moving on more general manifolds $\mathcal{M}$. The resulting action is a non-linear σ-model. The values of the background fields of this action are fixed by demanding conformal invariance of the quantum theory. This condition is satisfied if the β-function equations of the non-linear σ-model vanish. There exist an infinite number of solutions that satisfy this condition. The WZW models will be specially interesting in this context, since these are exactly solvable theories.

III. In chapter 3 we explain the SL(2,$\mathbb{R}$)/U(1) coset model, following closely [5,7]. This model can be interpreted as describing the propagation of strings in the black hole background. In section 3.1 we present the SL(2,$\mathbb{R}$)/U(1) gauged WZW model. We will see that after choosing the Lorentz gauge the quantization procedure of the theory gets simple. In section 3.2 we explain this quantization and review some basic facts about the representation theory of SL(2,$\mathbb{R}$), which is relevant to classify the physical states of the black hole CFT. The form of the physical states of the coset model can be determined with the BRST quantization procedure. This is explained in section 3.3. In section 3.4 we present Witten's semiclassical interpretation of the target space geometry as a 2D black hole, which has a similar structure as the Schwarzschild solution of Einstein's equations. In section 3.5 we will see how these results get corrections in $1/k$ as shown in the mini-superspace description of Dijkgraaf et al.

IV. In chapter 4 we are going to introduce the free-field representation of the black hole CFT. In section 4.1 we will see how the SL(2,$\mathbb{R}$) operator algebra of Kac-Moody currents can be realized through the Wakimoto representation of SL(2,$\mathbb{R}$), thus making it clear that the SL(2,$\mathbb{R}$) symmetry is manifest in the free-field approach. Using the Sugawara prescription we can construct the energy-momentum tensor and therefore the action in terms of these coordinates. We introduce the SL(2,$\mathbb{R}$) screening charge, that is an operator that guarantees the charge conservation of the correlation functions and added to the action is considered as the interaction of the model. Then it is simple to obtain the form of the gauged-fixed action and the form of the Kac-Moody primaries in terms of these coordinates. In section 4.2 we will see how this model can be obtained from the Lagrangian of the SL(2,$\mathbb{R}$) gauged WZW model choosing a concrete parametrization of the SL(2,$\mathbb{R}$)-valued field g in terms of the Gauss decomposition.

V. In chapter 5 we are going to make the space-time interpretation of the SL(2,$\mathbb{R}$)/U(1) gauged WZW model in terms of Wakimoto coordinates [30]. We are able to find the connection to the space-time interpretation of ref. [7]. This connection for $k = 9/4$ is important, since it is only in this case that we are able to evaluate the scattering amplitudes in the free-field approach, as we will see in the next chapters [32,30].

VI. To see which are the characteristics that the $c = 1$ model coupled to Liouville theory shares with the black hole CFT we present, as a starting point, the comparison between the cohomologies of the two models. In section 6.1 we write down the classification, of Distler and Nelson, of all the physical states that are allowed from the representation theory [22]. In section 6.2 we show how some of the discrete states of the previous classification look like in terms of Wakimoto coordinates. The simplest, new, discrete state that has no analogue in $c = 1$ coupled to Liouville turns out to be BRST-trivial [27]. The black hole mass operator is the discrete state $W^-_{1,0}\ \bar{W}^-_{1,0}$ of $c = 1$ [27,24]. A systematic analysis of the cohomology of the free-field model can be made through the isomorphism between the energy-momentum tensors of both theories as shown by Eguchi et al. [24]. This will be presented in section 6.3.

VII. In chapter 7 we consider the $\mathcal{S}$-matrix of tachyons interacting in the black hole background [32,30]. Using the free-field approach we are able to perform a direct computation of the correlation functions. In section 7.1 we formulate the problem and in section 7.2 we show how, after performing the zero mode integration of the fields, we are left with correlators of a free theory [27].

In section 7.3 we consider the correlation functions in the bulk; they are special since they satisfy the energy conservation [27]. These correlators have an obvious relation to the tachyon correlation functions in standard non-critical string theory. We then consider correlation functions where the number of screening charges is different from zero. In section 7.4 we begin with the correlation function containing one highest-weight state and can obtain the general three-point function using the SL(2,$\mathbb{R}$) Ward identities. As a result of our computation we observe that these correlators share a remarkable analogy with tachyon correlators of standard non-critical string theory. They factorize in leg factors that have poles in intermediate channels where all the discrete states belonging to the BRST cohomology of $c = 1$ are present. The new discrete states of Distler and Nelson do not appear. They are therefore BRST-trivial and decouple from the correlation functions. This means that the BRST cohomology of the 2D black hole is the one of $c = 1$, which is in agreement with the result of ref. [24]. The parameter M (that is related to the mass of the black hole) has to be renormalized in order to get non-vanishing correlation functions; this is a fact known from $c = 1$ matter coupled to Liouville theory perturbed by the cosmological constant, where the parameter μ is infinitely renormalized. The integrals that have to be evaluated are singular for $k = 9/4$, so that a careful treatment of the regularisation and renormalization is in order. In section 7.5 we calculate the three-point function with one screening as an illustrative example and analyse contact term interactions that arise in our computations in detail. To see whether the characteristics that appeared are generic for all the amplitudes, we compute the two-point function of (not necessarily) on-shell tachyons. Here a similar result is found, which indicates that the N-point functions might also have these characteristics. We explicitly compute the four-point tachyon amplitude with the chirality configuration $(+,+,+,-)$. We apply similar techniques as those used in ref. [34]; i.e. we compute the pole structure, the asymptotic behavior and the symmetries of the amplitude to determine its final form. The three-point function of on-shell tachyons is one of the basic ingredients.

VIII. Chapter 8 contains our conclusions and outlook.

We review the most important formulas for our calculations in the Appendix.

2. STRINGS MOVING ON BLACK HOLES

In this chapter we will introduce some basic notions in order to describe the propagation of strings in the black hole background. We first review shortly the classical solution to Einstein's equations. It is generally accepted that these equations have to be modified in a proper description in terms of quantum gravity. We will see how string theory modifies general relativity at a scale determined by the string coupling constant α', which is believed to be of the order of the Planck scale.

This short introduction is not supposed to be a complete review on the subject because this can be found in textbooks about gravitation and black holes [35] and about string theory [36]. Furthermore there exist some excellent review articles on the subject [2,37].

2.1 Black Holes in General Relativity

Nearly all the work done in general relativity before 1960 was concerned with solving the Einstein equation in a particular coordinate system. In the early 1970's there appeared the "no hair" theorems that state that a stationary black hole is uniquely determined by it's mass M, charge Q and its intrinsic angular momentum J. We are first going to see how the simplest black hole solution of Einstein's equations look like. It has no charge and no angular momentum. The gravitational field equations take in empty space the following form

$$\mathcal{R}_{ik} = 0. \tag{2.1}$$

A solution in four-dimensional space-time is the Schwarzschild metric represented by the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and (r, θ, ϕ) are the three-dimensional spherical coordinates. This is the unique spherically symmetric vacuum solution of Einstein's equation and is often used to represent the empty space region surrounding a spherical star or collapsing body of mass M. This metric appears to have a singularity at $r = 2M$. But, by looking at the curvature invariant

$$I = \mathcal{R}_{ijkl}\mathcal{R}^{ijkl} = 48\frac{M^2}{r^6}, \tag{2.3}$$

this point turns out to be just a coordinate singularity. It can be removed by a simple change of coordinates. There are many different coordinate transformations that can be done to show that $r = 2M$ is not a physical singularity. One of them is the so-called Kruskal coordinates:

$$\begin{aligned} \bar{u} &= -4M \exp\left(\frac{r^* - t}{4M}\right) \\ \bar{v} &= 4M \exp\left(\frac{r^* + t}{4M}\right). \end{aligned} \tag{2.4}$$

Here r^* represents the tortoise coordinate

$$r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right), \tag{2.5}$$

which satisfies $dr = (1 - 2M/r)dr^*$. The line element can be written in terms of these coordinates in the following form

$$ds^2 = -\frac{2M}{r}e^{-r/2M}d\bar{u}d\bar{v} + r^2 d\Omega^2. \tag{2.6}$$

Clearly this metric is non-singular at $r = 2M$. However, $r = 0$ is still a singularity, as can be seen by comparing with the form of the scalar curvature (2.3). This is a point with an infinite gravitational field strengths.

The original Schwarzschild coordinate system covers only part of the space-time manifold. The region $r \geq 2M$ corresponds to $-\infty < \bar{u} < 0$ and $0 < \bar{v} < \infty$. In Kruskal coordinates we can analytically continue this solution to the whole region $-\infty < \bar{u}, \bar{v} < \infty$. The resulting Kruskal diagram is an extension of the Schwarzschild black hole. It consists of several different regions. Region I represents the region outside the horizon and is asymptotically flat. The horizons of the black hole are given by the two lines $\bar{u} = 0$ and $\bar{v} = 0$. The physical singularity is located at $\bar{u}\bar{v} = 1$, so that it has the form of an hyperbola. Region III is the region between the black hole singularity and the horizon. Once an observer has entered this region he can (classically) never escape from it. Region IV has the "time-reversed" properties of region III and is called a white hole. Region II is asymptotically flat, with $r \geq 2M$. Region V and VI are the regions of negative mass above and below the singularities.

Fig. 1: Kruskal diagram of the Schwarzschild metric.

It is important to realize that the full analytically continued Schwarzschild metric is merely a mathematical solution of Einstein's equations. For a black hole formed by gravitational collapse, part of the space-time must contain the collapsing matter. The Schwarzschild metric is a solution of the gravitational field equations in empty space. The outside of a collapsing star is still described by the Schwarzschild metric. Thus the world-line of a point on the surface of the star will be the boundary of the physically meaningful part of the Kruskal diagram. The "white hole" and the regions II, IV, V and VI are not present in real black holes.

A more general black hole solution to Einstein's equations that is characterized by (M, J, Q) is called the Kerr-Newman black hole. A special case of this solution is the Reissner-Nordström black hole, which has no angular momentum $J = 0$. It is characterized by the line element

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \qquad (2.7)$$

This space-time has a curvature singularity at $r = 0$ as for Schwarzschild but it has in addition two horizons, where $1 - 2M/r + Q^2/r^2$ vanishes:

$$r_\pm = M \pm (M^2 - Q^2)^{1/2}. \qquad (2.8)$$

The extremal Reissner-Nordström black hole corresponds to the case $M = Q$ and plays a special role in connection with Witten's 2D black hole solution and the $c = 1$ matrix model [5,10,11].

2.2 Strings in Curved Backgrounds

To address the connection between string theory, singularities and strong gravitational fields, it is important to study strings in curved backgrounds. We are going to see that WZW models naturally appear in this context.

The propagation of a string in flat Minkowski space-time is described by the action:

$$\mathcal{S}_0 = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}, \tag{2.9}$$

where $h_{\alpha\beta}$ is the world-sheet metric which, is regarded, in a string theory as a dynamical variable and $\eta_{\mu\nu}$ is the Minkowski metric. The parameter α' is the string coupling constant that is a free parameter of dimension $(length)^2$ that makes the expressions dimensionless. It plays the role of Planck's constant and the classical limit corresponds to small α'. Quantum mechanical perturbation theory is therefore an expansion in α'. The variables $X^\mu(\sigma)$, where $\mu = 1, \ldots, d$, are scalar fields. The above action is known as the Polyakov action. It is classically invariant under the reparametrizations of the string world-sheet $\sigma \to \sigma'$ and it has a local Weyl symmetry or conformal symmetry that is manifest through the vanishing of the classical energy-momentum tensor:

$$T_{\alpha\beta} = -\frac{2\pi}{\sqrt{h}}\frac{\delta\mathcal{S}_0}{\delta h^{\alpha\beta}} = 0. \tag{2.10}$$

This equation is known as the Virasoro condition. If we would like to consider string propagation on an arbitrary manifold we would have to consider the following, more general, action called a non-linear σ-model:

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\mathcal{G}_{\mu\nu}(X). \tag{2.11}$$

As in the flat space-time, this action is invariant under reparametrizations of the world-sheet coordinates σ and it has a classical local Weyl invariance. In a more systematic way one can include all of the massless states of the closed string (and not only the graviton) as part of the background [36]. The action takes then the following form

$$
\begin{aligned}
\mathcal{S} = &- \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} (h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \mathcal{G}_{\mu\nu}(X) + T(X)) \\
&- \frac{1}{4\pi\alpha'} \int d^2\sigma \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu \mathcal{B}_{\mu\nu}(X) + \frac{1}{4\pi} \int d^2\sigma \sqrt{h} R \Phi(X).
\end{aligned}
\tag{2.12}
$$

The above action is a functional of the world-sheet metric $h_{\alpha\beta}$ and the d space-time coordinates X^μ. The previous ansatz for the action is the most general one, involving only renormalizable interactions. The couplings $\mathcal{G}_{\mu\nu}(X)$, $\Phi(X)$, $T(X)$ and $\mathcal{B}_{\mu\nu}(X)$ can be identified with the graviton, dilaton, tachyon and the antisymmetric tensor respectively. Their values are restricted by demanding conformal invariance or local scale invariance of the theory. The best way to impose this condition is to consider the theory in $2+\epsilon$ dimensions and to calculate those terms of the action that break the symmetry at the quantum level in the limit $\epsilon \to 0$. Demanding these terms to vanish restricts the values of the couplings of the theory. These constraints can be formulated in terms of the so-called β-functions. There is one β-function for each of the fields $\mathcal{O}_i$, and the trace of the energy-momentum tensor is formulated in terms of these functions

$$
\langle T_{z\bar{z}} \rangle = \int e^{-\mathcal{S}} \beta_i \mathcal{O}^i. \tag{2.13}
$$

Therefore the statement of conformal invariance translates into the requirement that the β-functions associated with the background fields vanish. These β-functions can be calculated using background field perturbation theory, i.e. as an expansion in α', but no closed expression is known that holds to all orders in the string coupling constant. To first order in α' these equations have the following form [38]:

$$\beta^{\mathcal{G}}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{4} H_\mu^{\lambda\sigma} H_{\nu\lambda\sigma} + 2\nabla_\mu\nabla_\nu\Phi - \nabla_\mu T \nabla_\nu T + O(\alpha') = 0$$

$$\beta^\Phi = \frac{d-26}{48\pi^2} - \frac{\alpha'}{16\pi^2}\left(4(\nabla\Phi)^2 - 4\nabla^2\Phi - \mathcal{R} + \frac{1}{12}H^2 + (\nabla T)^2 + V(T)\right) + O(\alpha'^2) = 0$$

$$\beta^T = -2\nabla^2 T + 4\nabla\Phi\nabla T + V'(T) + O(\alpha') = 0$$

$$\beta^{\mathcal{B}}_{\mu\nu} = \nabla_\lambda H^\lambda_{\mu\nu} - 2(\nabla_\lambda\Phi)H^\lambda_{\mu\nu} + O(\alpha') = 0. \tag{2.14}$$

Here $\mathcal{R}$ is the curvature of $\mathcal{G}_{\mu\nu}$, $V(T) = -2T^2 + O(T^3)$ is the tachyon potential and $H_{\mu\nu\lambda} = \nabla_\mu \mathcal{B}_{\nu\lambda} + \nabla_\lambda \mathcal{B}_{\mu\nu} + \nabla_\nu \mathcal{B}_{\lambda\mu}$ is the antisymmetric tensor field strength.

Combining the β-functions for T and Φ it is possible to write [38,39]:

$$\beta^{\mathcal{G}}_{\mu\nu} + 8\pi^2\mathcal{G}_{\mu\nu}\frac{\beta^\Phi}{\alpha'} = \left(\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{G}_{\mu\nu}\mathcal{R}\right) - T^{matter}_{\mu\nu} = 0 \tag{2.15}$$

where

$$T^{matter}_{\mu\nu} = \frac{1}{4}\left[H^2_{\mu\nu} - \frac{1}{6}\mathcal{G}_{\mu\nu}H^2\right] - 2\nabla_\mu\nabla_\nu\Phi + 2\mathcal{G}_{\mu\nu}\nabla^2\Phi - 2\mathcal{G}_{\mu\nu}(\nabla\Phi)^2. \tag{2.16}$$

This is the Einstein equation for the background metric $\mathcal{G}_{\mu\nu}$. However, if we include two-loop corrections we have to add a term of the form

$$\frac{\alpha'}{2}\mathcal{R}_{\mu\lambda\sigma\tau}\mathcal{R}_{\nu}^{\lambda\sigma\tau}, \tag{2.17}$$

to equation (2.15), which gives us corrections to the Einstein equation coming from string theory.

Conformal invariance imposes strong constraints on the quantum theory, but it does not fix the theory uniquely. Two solutions of the β-function equations that are believed to be exact i.e. to all orders in α' are Witten's eternal black hole [5] and the Liouville solution in flat space found by David [40] and Distler and Kawai (DDK) [41]. The latter has the following form

$$\begin{aligned} &T = \frac{\mu}{2\gamma^2}e^{\gamma\phi} \qquad \text{with} \qquad Q = \frac{2}{\gamma} + \gamma = \sqrt{\frac{26-d}{3}} \\ &\Phi = \frac{Q}{2}\phi \\ &\mathcal{G}_{\mu\nu} = \delta_{\mu\nu} \end{aligned} \tag{2.18}$$

where $\phi = X_0$. These expressions solve the lowest order in α' of the β-function equations if the lowest order in T is taken into account but they are not a solution if higher powers of T are considered. One might hope that higher orders in α' of the β-functions correct this problem and that probably after a redefinition of the fields the above solution is exact [42].

Inserting these expressions into the σ-model action (2.12), we get the familiar expression for the Liouville action

$$\mathcal{S} = \frac{1}{8\pi}\int d^2\sigma\sqrt{\widehat{h}}\left(\widehat{h}^{ab}\partial_a\phi\partial_b\phi + Q\widehat{R}\phi + \frac{\mu}{\gamma^2}e^{\gamma\phi}\right), \tag{2.19}$$

where we have considered the analytic continuation to the Euclidean theory with

$\alpha' = 2$ and we have used the conformal gauge $h = \widehat{h}e^{\gamma\phi}$. Equations (2.14) can be derived as Euler-Lagrange equations of the following effective action

$$\mathcal{S}_{eff} = \int d^2\sigma\sqrt{\mathcal{G}}e^{-2\Phi}((\nabla T)^2 - 2T^2 - 4(\nabla\Phi)^2 - \frac{1}{12}H^2 - \mathcal{R} + (d-26)/3 + \ldots). \quad (2.20)$$

If we consider weak field tachyons for which we can surely neglect higher powers of T, the equation of motion of this field can be written in the form of a Klein-Gordon equation

$$(L_0 - 1)T = 0, \quad (2.21)$$

where L_0 is given by

$$L_0 = -\frac{1}{2e^{\Phi}\sqrt{\mathcal{G}}}\partial_i e^{\Phi}\sqrt{\mathcal{G}}\mathcal{G}^{ij}\partial_j. \quad (2.22)$$

This equation will be used later to identify the background metric $\mathcal{G}_{\mu\nu}$ and the dilaton Φ of the SL(2,$\mathbb{R}$)/U(1) WZW model [7].

The low-energy effective action without the tachyon contribution describes a $(1+1)$-dimensional black hole toy model known as dilaton gravity [4]. Without coupling this model to additional matter it is equivalent to the lowest order in α' of the black hole that was first studied by Witten [5]. However, when we couple this model to N scalar fields one can study the formation of a 2D black hole and the process of Hawking radiation.

Corrections to the low-energy effective action can be computed by calculating higher-order corrections to the β-functions and can involve, for example, higher derivative terms or higher powers of the tachyon. We will mainly be concerned with the exact solution of the β-function equation that describes Witten's black hole. It has been found in ref. [5] to be described in terms of a gauged WZW model.

Naively, the connection between the previous σ-model and the WZW model can be motivated in the following way. Solving the model described by the action (2.11) is in general very difficult, so that one has to make simplifications for the background metric. One of these simplifications is to assume that the string propagates on a group manifold of a semisimple Lie group G. If g is an element of G, then in analogy to the σ-model the first guess for the action in terms of g would be [43,44,45,46]

$$\mathcal{S} = \int_{\Sigma} \mathrm{tr}(\partial_a g^{-1} \partial^a g) d^2\xi. \tag{2.23}$$

The field g is some function of the string field X_μ in terms of which we can express the metric $\mathcal{G}_{\mu\nu}$. This naive choice of the action needs corrections because it is not conformal-invariant. We can modify the action

$$\mathcal{S} = \frac{1}{4\lambda^2} \int \mathrm{tr}(\partial_a g^{-1} \partial^a g) d^2\xi + ik\Gamma(g), \tag{2.24}$$

by adding the so-called Wess-Zumino term

$$\Gamma(g) = \frac{1}{24\pi} \int d^3X \epsilon^{\alpha\beta\gamma} \mathrm{tr}[(g^{-1}\partial_\alpha g)(g^{-1}\partial_\beta g)(g^{-1}\partial_\gamma g)]. \tag{2.25}$$

In the above formula we integrate over a three-dimensional manifold with boundary equal to Σ. The characteristics of the resulting action are very different from those of the action with $k = 0$. It represents a conformal-invariant σ-model for special values of λ:

$$\lambda^2 = \frac{4\pi}{k}. \tag{2.26}$$

This action is called the Wess-Zumino-Witten model.

3. THE SL(2,$\mathbb{R}$)/U(1) GAUGED WZW MODEL

In this chapter we are going to see how an exact conformal field theory, the SL(2,$\mathbb{R}$)/U(1) gauged WZW model, can be regarded as a model describing the propagation of strings in the black hole background. We review the Lagrangian formulation of the SL(2,$\mathbb{R}$)/U(1)conformal field theory and present the semiclassical approach of Witten [5], which relates the background of the WZW model to a Schwarzschild-like space-time. We will explain the mini-superspace approach of Dijkgraaf, H. Verlinde and E. Verlinde [7]. Here it will be clear how the semiclassical metric found by Witten gets corrections of order $1/k$.

3.1 Lagrangian Formulation and Gauge Fixing

The conformal field theory that describes a black hole in two-dimensional target space-time has a Lagrangian formulation in terms of a gauged WZW model based on the non-compact group SL(2,$\mathbb{R}$) [5]. The ungauged SL(2,$\mathbb{R}$) WZW model is described by following action:

$$S_{WZW}(g) = \frac{k}{8\pi} \int_{\Sigma} d^2x \sqrt{h} h^{ij} \, \mathrm{tr}(g^{-1}\partial_i g g^{-1}\partial_j g) + ik\Gamma(g), \tag{3.1}$$

where Σ is a Riemann surface with metric tensor h^{ij}, $g : \Sigma \to$SL(2,$\mathbb{R}$) is an SL(2,$\mathbb{R}$)-valued field on Σ and k is a real and positive number. The Wess-Zumino term, which guarantees conformal invariance, is represented by:

$$\Gamma(g) = \frac{1}{12\pi} \int_{B} d^3y \varepsilon^{abc} \, \mathrm{tr}(g^{-1}\partial_a g g^{-1}\partial_b g g^{-1}\partial_c g), \tag{3.2}$$

where B is a three-dimensional manifold with boundary Σ.

The action (3.1) possesses a global SL(2,$\mathbb{R}$)$\times$SL(2,$\mathbb{R}$) symmetry, since it is invariant under $g \to agb^{-1}$ with $a, b \in$SL(2,$\mathbb{R}$). The reason is that the products that appear in the action are of the form

$$g^{-1}\partial g \rightarrow b g^{-1}\partial g b^{-1} \tag{3.3}$$

and the trace is clearly invariant under this change of basis. To get the interpretation as a 2D black hole we are interested in the gauging of a subgroup of this symmetry group. Depending on which subgroup we gauge, we get the Euclidean version of the black hole or its Lorentzian counterpart. These solutions can also be obtained as analytic continuation from one to the other. To get the Euclidean version of the black hole we gauge an Abelian subgroup h of SL(2,$\mathbb{R}$) that is compact, while for the Minkowski version this subgroup is non-compact. Therefore, in the Euclidean theory we set

$$a = b^{-1} = h = \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}. \tag{3.4}$$

For ϵ small we can represent $h = 1 + \epsilon G$, where

$$G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.5}$$

Under the transformation $g \rightarrow hgh$ the field g is not invariant but transforms as

$$\delta g = \epsilon(Gg + gG). \tag{3.6}$$

To show that the action (3.1) is invariant under the above transformation, we have to assume that the parameter ϵ is not dependent on the coordinates, i.e. that we have a global symmetry. If we would like to make this symmetry local we must introduce a gauge field A that satisfies $\delta A_i = -\partial_i\epsilon$. The gauge-invariant generalization of the WZW action then takes the following form:

$$S_{WZW}(g,A) = S_{WZW}(g) + \frac{k}{2\pi}\int d^2z(\bar{A}\mathrm{tr}(Gg^{-1}\partial g) + A\mathrm{tr}(G\bar{\partial}gg^{-1}) + A\bar{A}(-2+\mathrm{tr}(GgGg^{-1}))). \tag{3.7}$$

Now we can choose a gauge. If we impose the Lorentz gauge condition $\partial_\alpha A^\alpha = 0$, the gauge slice can be parametrized as $A^\alpha = \varepsilon^{\alpha\beta}\partial_\beta X$ or

$$A = \partial X \qquad \text{and} \qquad \bar{A} = -\bar{\partial}X. \tag{3.8}$$

The complete gauge-fixed action of the Euclidean theory is then given by [7]:

$$S^{gf}_{WZW} = S_{WZW}(g) + S(X) + S(B,C). \tag{3.9}$$

Here X is a free scalar field:

$$S(X) = \frac{k}{4\pi}\int d^2z \partial X\bar{\partial}X, \tag{3.10}$$

which for the Euclidean theory is compact, with compactification radius $R = \sqrt{k}$ in units of the self-dual radius. The fields (B,C) are a spin $(1,0)$ system of fermionic ghosts that come from the Jacobian of the redefinition (3.8):

$$S(B,C) = \int d^2z(B\bar{\partial}C + \bar{B}\partial\bar{C}). \tag{3.11}$$

This form for the action can be proved either using an explicit representation in terms of Euler angles [7], as we will see later or more generally as done by Gawedzki and Kupiainen [47] without using an explicit representation for g. From (3.9) we see that the gauged WZW model can be expressed through the ungauged SL(2,$\mathbb{R}$) WZW model and the action of the free-fields X, B and C. As already remarked in ref. [7] this makes the quantization straightforward.

3.2 Current Algebra and the Representation Theory of SL(2,$\mathbb{R}$)

To quantize the ungauged theory [7] one notices that the SL(2,$\mathbb{R}$) symmetry gives rise to the conserved currents $\partial\bar{J} = 0$ and $\bar{\partial}J = 0$:

$$J = J^a t^a = -\frac{k}{2}\partial g g^{-1}, \qquad\qquad \bar{J} = \bar{J}^a t^a = -\frac{k}{2}g^{-1}\bar{\partial}g, \tag{3.12}$$

where $t^1 = i\sigma_1/2$, $t^2 = \sigma_2/2$ and $t^3 = i\sigma_3/2$ are the generators of SL(2,$\mathbb{R}$) and σ_i are the Pauli matrices. The above form of the currents follows as equations of motion of the action (3.1), considering the variation δS_{WZW} under the transformation $g \to g + \delta g$. The modes of these currents satisfy the SL(2,$\mathbb{R}$) current algebra of level k, which is equivalent to the following operator product expansion (OPE):

$$\begin{aligned} J_+(z)J_-(w) &= \frac{k}{(z-w)^2} - \frac{2J_3(w)}{(z-w)} + \dots \\ J_3(z)J_\pm &= \pm\frac{J_\pm(w)}{(z-w)} + \dots \\ J_3(z)J_3(w) &= -\frac{\frac{k}{2}}{(z-w)^2} + \dots, \end{aligned} \tag{3.13}$$

which, after expanding in modes, can be written as

$$\begin{aligned} [J^3_n, J^3_m] &= -\frac{1}{2}kn\delta_{n+m,0} \\ [J^3_n, J^\pm_m] &= \pm J^\pm_{n+m} \\ [J^+_n, J^-_m] &= kn\delta_{n+m,0} - 2J^3_{n+m}. \end{aligned} \tag{3.14}$$

Here we have introduced the notation $J_+ = J_1 + iJ_2$ and $J_- = J_1 - iJ_2$.

The representation theory of Kac-Moody algebras shares many features with the Virasoro algebra. In the following we are going to explain some basic notions about the representation theory of SL(2,$\mathbb{R}$) [48,49] that we will need later. The basic fields from which we can build all the other states are the Kac-Moody primaries that satisfy:

$$J_n^{\pm}|j,m\rangle = J_n^3|j,m\rangle = 0 \qquad \text{for} \quad n>0, \tag{3.15}$$

and are characterized by the zero-mode Casimir eigenvalue j and by the eigenvalue of J_0^3:

$$\Delta_0|j,m\rangle = j(j+1)|j,m\rangle, \qquad\qquad J_0^3|j,m\rangle = m|j,m\rangle, \tag{3.16}$$

where $\Delta_0 = \frac{1}{2}(J_0^+J_0^- + J_0^-J_0^+) - (J_0^3)^2$. Here we have introduced a holomorphic notation, but the same holds for the antiholomorphic part. We can construct the representation by acting with raising and lowering operators J_0^+ and J_0^-, as we know for the ordinary harmonic oscillator. A solution of (3.16) is

$$\begin{aligned} J_0^+|j,m\rangle &= (-m+j)|j,m+1\rangle \\ J_0^-|j,m\rangle &= (-m-j)|j,m-1\rangle. \end{aligned} \tag{3.17}$$

If $j \in \mathbb{R}$ it is standard in the representation theory of SL(2,$\mathbb{R}$) [49] to introduce new states $|j,m)$ that satisfy

$$J_0^+|j,m) = \sqrt{(j-m)(j+m+1)}|j,m+1)$$

$$J_0^-|j,m) = \sqrt{(j+m)(j-m+1)}|j,m-1) \tag{3.18}$$

$$J_0^3|j,m) = m|j,m).$$

These recursion relations are satisfied up to a function depending only on j, if we normalize the states as:

$$|j,m\rangle = \sqrt{\Gamma(j+m+1)\Gamma(j-m+1)}|j,m). \tag{3.19}$$

We will later work with the states that satisfy (3.17) unless otherwise stated. These Kac-Moody primaries define an irreducible representation of SL(2,$\mathbb{R}$), on which we can impose two types of constraints [22]:

1. *Hermiticity constraints.* We demand that Δ_0 and J_0^3 should have real eigenvalues. This means $m \in \mathbb{R}$ and $j = -\frac{1}{2} + i\lambda$ or $j \in \mathbb{R}$. The types of Hermitian representations of SL(2,$\mathbb{R}$) can be classified as follows [50]:

- Principal discrete series: Highest-weight or lowest-weight representation. Contain a state annihilated by ${J_0}^+$ and ${J_0}^-$, respectively. They are one-sided and infinite-dimensional. From (3.17) we see that these modules satisfy either $(j+m)$ or $(j-m)$ is an integer and:

$$\begin{aligned} |\text{HWS module}\rangle &= |j,m\rangle \qquad m = j, j-1, \ldots \\ |\text{LWS module}\rangle &= |j,m\rangle \qquad m = -j, -j+1, \ldots \end{aligned} \tag{3.20}$$

  If in addition $2j$ is an integer, the representation is double-sided.

- Principal continuous series. Satisfies $j = -\frac{1}{2} + i\lambda$ with λ, $m \in \mathbb{R}$.

- Supplementary series. In this case $j \in \mathbb{R}$, but neither $j+m$ nor $j-m$ is an integer.

2. *Unitarity constraints.* The states are constrained to have positive norm, i.e. $J_0^+ J_0^-$ and $J_0^- J_0^+$ should have positive eigenvalues. This imposes restrictions on the allowed values of j. We will not impose any constraints of unitarity on our states.

We will later see that on-shell states of the Euclidean black hole belong to the discrete and supplementary series, while the principal continuous series is off-shell. In the Minkowski theory the on-shell states are those corresponding to the principal continuous series.

We can create the above states by acting with a vertex operator $T_{j\,m}$ on the SL(2)-invariant vacuum

$$|j, m\rangle = \lim_{z \to 0} T_{j\,m}(z)\, |0\rangle. \tag{3.21}$$

These vertex operators have the following OPE with the energy-momentum tensor and the currents:

$$\begin{aligned} T(z) T_{j\,m}(w) &= \frac{h_{j,m}}{(z-w)^2} T_{j\,m}(w) + \frac{1}{(z-w)} \partial T_{j\,m}(w) + \ldots \\ J^a(z) T_j(w) &= \frac{t^a_{(j,m)}}{(z-w)} T_j(w) + \ldots \end{aligned} \tag{3.22}$$

Where by T_j we mean the multiplet of states with fixed j. The correlation functions of these vertex operators satisfy the SL(2,$\mathbb{R}$) Ward identities that have the following form:

$$\langle T(z)T_1(z_1,\bar{z}_1)\dots T_N(z_N,\bar{z}_N)\rangle = \sum_{i=1}^{N}\left(\frac{h_i}{(z-z_i)^2}+\frac{1}{(z-z_i)}\frac{\partial}{\partial z_i}\right)\langle T_1(z_1,\bar{z}_1)\dots T_N(z_N,\bar{z}_N)\rangle$$

$$\langle J^a(z)T_{j_1}(z_1,\bar{z}_1)\dots T_{j_N}(z_N,\bar{z}_N)\rangle = \sum_{i=1}^{N}\frac{t_i^a}{(z-z_i)}\langle T_{j_1}(z_1,\bar{z}_1)\dots T_{j_N}(z_N,\bar{z}_N)\rangle. \tag{3.23}$$

They can be derived by pushing a contour integral through the correlator, where the contour encloses each of the points z_i. Deforming the contour to a sum over small contours (each of them enclosing one of the z_i's), we get the above result.

Given a Kac-Moody algebra we can always construct a corresponding enveloping Virasoro algebra. The stress tensor follows from the Sugawara construction and is given by the following expression:

$$T_{\mathrm{SL}(2,\mathbb{R})} = -\frac{\Delta}{(k-2)}, \qquad \Delta = -\frac{1}{2} : J_+J_- + J_-J_+ : + : J_3J_3 : . \tag{3.24}$$

The above prescription is a generalization of the U(1) case, where the current is $j = \partial\phi(z)$ and the energy-momentum tensor is given by $T(z) = -\frac{1}{2} : j(z)j(z) :$. The normalization constant $1/(k-2)$ in the expression (3.24) is fixed by requiring that the currents are indeed $(1,0)$ primary fields. The modes of the stress tensor can be expressed through the currents

$$L_n = -\frac{1}{k-2}\sum_{m=-\infty}^{\infty} : J^a_{m+n}J^a_{-m} : .$$

These currents have the following OPE with the energy-momentum tensor

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)} + \dots, \tag{3.25}$$

that is equivalent to the commutation relation

$$[L_m, J^a_n] = -nJ^a_{m+n}. \tag{3.26}$$

The stress tensor has a central charge as a function of the level k of the Kac-Moody algebra:

$$c = \frac{3k}{k-2}, \tag{3.27}$$

that can be obtained from the OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \dots. \tag{3.28}$$

Using the previously introduced OPEs, it is also easy to deduce that the conformal weight $h_{j,m}$ of a primary field field $T_{j\ m}$ is given by

$$h_{j,m} = -\frac{j(j+1)}{k-2}. \tag{3.29}$$

In the case of the SL(2,$\mathbb{R}$)/U(1) coset model the energy-momentum tensor that follows from the gauged fixed action is

$$T_{\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)} = -\frac{\Delta}{(k-2)} - \frac{1}{2}(\partial X)^2 - B\partial C, \tag{3.30}$$

which has a central charge

$$c = \frac{3k}{k-2} - 1. \tag{3.31}$$

There appear two interesting regions. The limit $k \to \infty$ corresponds to the semi-classical limit in which the σ-model is weakly coupled. From (3.7) it is clear that k plays the role of $\hbar^{-1}$ in the quantum theory. If we choose $k = 9/4$ the central charge will be $c = 26$ and the theory describes a critical bosonic string in a curved background.

The total central charge of the theory is zero if we take into account the reparametrization ghosts. This is a fermionic system of ghosts described by the action

$$S(b,c) = \int d^2z (b\bar{\partial}c + \bar{b}\partial\bar{c}). \tag{3.32}$$

The field b has conformal dimensions $(h,\bar{h}) = (2,0)$, c has dimension $(h,\bar{h}) = (-1,0)$, and the two-point function of these fields is

$$\langle b(z)c(w)\rangle = \langle c(z)b(w)\rangle = \frac{1}{(z-w)}. \tag{3.33}$$

The energy-momentum tensor of this ghost system has a central charge of $c = -26$ and is given by

$$T^{b,c}(z) = -2b(z)\partial c(z) - \partial b(z)c(z). \tag{3.34}$$

Gauged WZW models are the Lagrangian formulation of coset CFTs. Such a coset construction can be associated to any current algebra. Given a symmetry group G and a subgroup H of G, the stress tensor of the coset CFT is then the difference of the two stress tensors

$$T_{G/H} = T_G - T_H. \tag{3.35}$$

Each of these operators can be constructed using the corresponding currents and the Sugawara prescription. The central charge of the coset theory is the difference between the two central charges:

$$c_{G/H} = c_G - c_H. \tag{3.36}$$

The connection between this coset construction and the gauged WZW models has been explained in [47,51].

3.3 Physical States of the Coset Model and BRST Quantization

In general, to obtain the form of the physical states of the quantum theory we can use the BRST quantization procedure. The BRST symmetry is a symmetry of the gauged-fixed action. Associated with this symmetry we have a nilpotent charge Q_{BRST} that commutes with the energy-momentum tensor

$$\delta T = [Q_{\text{BRST}}, T] = 0. \tag{3.37}$$

A state is said to be BRST-invariant if it is annihilated by the BRST charge. This is a necessary condition for gauge invariance and should therefore be satisfied by physical states. A state $|\phi\rangle$ that satisfies

$$|\phi\rangle = Q_{\text{BRST}}|\phi'\rangle$$

is said to be BRST-trivial. It is annihilated by Q_{BRST} because of the nilpotency of this operator, and it decouples in $\mathcal{S}$-matrix elements. Therefore, physical states have to be BRST-invariant but not trivial. This BRST invariance is equivalent to the condition

$$[Q_{\mathrm{BRST}}, \phi(z)] = \text{total derivative.} \tag{3.38}$$

In the case of the black hole CFT, we have two BRST charges corresponding to the U(1) symmetry and the diffeomorphism invariance that are called $Q^{\mathrm{U}(1)}$ and Q^{Diff} respectively. Their expressions are given by

$$Q^{\mathrm{U}(1)} = \oint C(z)\Big(J^3 - i\sqrt{\frac{k}{2}}\partial X\Big)dz, \tag{3.39}$$

and

$$Q^{Diff} = \oint c(z)\Big(T_{\mathrm{SL}(2,\mathrm{IR})/\mathrm{U}(1)} + \frac{1}{2}T_{gh}\Big). \tag{3.40}$$

If we would like to consider the anti-holomorphic BRST-constraints, we have to take into account that there are two possible choices of sign for the current

$$\bar{J}_3^{total} = \bar{J}_3 \pm i\sqrt{\frac{k}{2}}. \tag{3.41}$$

Both choices are related by a duality transformation.

It is easy to verify that $T_{\mathrm{SL}(2,\mathrm{IR})/\mathrm{U}(1)}$ commutes with each of the two BRST charges. These operators satisfy

$$(Q^{Diff})^2 = 0, \qquad \{Q^{\mathrm{U}(1)}, Q^{Diff}\} = 0. \tag{3.42}$$

If we consider states of ghost number zero, the condition (3.38) is equivalent up to a total derivative to the physical on-shell condition, which demands the physical state to have dimension $(1,1)$. More explicitly we see that the expression

$$\begin{aligned}\left[Q^{Diff},\phi(z)\right] &= \oint \frac{dw}{2\pi i} c(w)T(w)\phi(z) \\ &= \oint \frac{dw}{2\pi i} c(w)\left(\frac{h\phi(z)}{(w-z)^2}+\frac{\partial\phi(z)}{(w-z)}+\dots\right) \\ &= h(\partial c)\phi(z)+c\partial\phi(z)\end{aligned} \tag{3.43}$$

is a total derivative if $h = 1$. The physical-state condition can be equivalently formulated as the following two equations:

$$\begin{aligned}(L_0+\bar{L}_0-2)|\phi\rangle &= 0 \\ (L_0-\bar{L}_0)|\phi\rangle &= 0.\end{aligned} \tag{3.44}$$

The invariance under the two BRST charges implies that physical states of the coset theory are of the form

$$\mathcal{V}_{j\ m\ \bar{m}} = T_{j\ m\ \bar{m}} \exp\left(i\sqrt{\frac{2}{k}}\left(mX(z)+\bar{m}\bar{X}(\bar{z})\right)\right), \tag{3.45}$$

where $T_{j\ m\ \bar{m}}$ are the fields of the ungauged theory. The quantities m and $\bar{m}$ denote the eigenvalues of the zero modes of the currents $J_3(z)$ and $\bar{J}_3(\bar{z})$ and we have used the notation $X(z,\bar{z}) = X(z)+\bar{X}(\bar{z})$. It is easy to verify that the above field is invariant under $Q^{\mathrm{U}(1)}$. The invariance under Q^{Diff} up to a total derivative implies the on-shell condition

$$h_{j,m} = -\frac{j(j+1)}{k-2}+\frac{m^2}{k} = 1; \tag{3.46}$$

for $k = 9/4$ this condition takes the form:

$$2j+1=\pm\frac{2}{3}m, \tag{3.47}$$

and the same for the antiholomorphic component.

In the Minkowski black hole the scalar field $X(z,\bar{z})$ is uncompactified, so that we have the restriction $m=\bar{m}$. For the Euclidean theory this field is compactified. This implies that the eigenvalues m and $\bar{m}$ have to be on the lattice [36]:

$$m=\frac{1}{2}(n_1+n_2k), \qquad \bar{m}=-\frac{1}{2}(n_1-n_2k), \qquad n_1,n_2\in\mathbb{Z}. \tag{3.48}$$

States with $n_1=0$ are called winding modes, while states with $n_2=0$ are called momentum modes. The spectrum of the Euclidean black hole CFT is therefore given by

$$h^j_{n_1n_2}=-\frac{j(j+1)}{k-2}+\frac{(n_1+n_2k)^2}{4k}, \tag{3.49}$$

$$\bar{h}^j_{n_1n_2}=-\frac{j(j+1)}{k-2}+\frac{(n_1-n_2k)^2}{4k}. \tag{3.50}$$

This coincides with the spectrum of a Liouville field with momenta $\alpha=\sqrt{\frac{2}{k-2}}j$ and a background charge $Q=\frac{2}{k-2}$ coupled to a scalar field with compactification radius

$$R=\sqrt{k}R_0, \tag{3.51}$$

where $R_0=1/\sqrt{2}$ is the self dual radius.

We will discuss the cohomology of the 2D black hole CFT in more detail in chapter 5. To continue we will choose a concrete representation for the SL(2,$\mathbb{R}$)-valued field g or equivalently for the currents. We will now see why this model describes strings in a black hole background.

3.4 Witten's Semiclassical Interpretation as a 2D Black Hole

In Witten's semiclassical description of the background, the gauge field A of the gauge-invariant action (3.7) is integrated out, using its equations of motion. This procedure is of course only valid in the semiclassical limit and we expect to get corrections in $1/k$.

We will first consider the Euclidean theory. In this case gauge invariance is fixed by setting

$$g = \cosh r + \sinh r \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{3.52}$$

The resulting action then takes the form:

$$\begin{aligned} S(r,\theta) &= \frac{k}{2\pi}\int d^2z(\partial r\bar{\partial}r + \tanh^2 r\partial\theta\bar{\partial}\theta) \\ &= \frac{k}{4\pi}\int d^2x\sqrt{h}h^{ij}(\partial_i r\partial_j r + \tanh^2 r\partial_i\theta\partial_j\theta). \end{aligned} \tag{3.53}$$

The Wess-Zumino term is a total derivative in this gauge and it has been dropped. By comparing (3.53) with (2.11) the target space metric $\mathcal{G}_{\mu\nu}$ can be computed. It is given by

$$\mathcal{G}_{\mu\nu} = \begin{pmatrix} \mathcal{G}_{rr} & \mathcal{G}_{r\theta} \\ \mathcal{G}_{\theta r} & \mathcal{G}_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tanh^2 r \end{pmatrix}. \tag{3.54}$$

We therefore obtain the following line element

Fig. 2: Target space geometry of the Euclidean black hole in the semiclassical limit (analogue to regions I and II of the Kruskal diagram of the Minkowskian Schwarzschild metric).

$$ds^2 = \mathcal{G}_{\mu\nu} dX^\mu dX^\nu = dr^2 + \tanh^2 r d\theta^2. \tag{3.55}$$

This metric corresponds to a surface of a semi-infinite cigar (see Fig. 2) that, in the asymptotic region $r \to \infty$, gets $\mathbb{R} \times S^1$.

From the measure in the integration over A there appears a finite correction that gives rise to the target space dilaton, so that the classical action has the form

$$\begin{aligned} S(r,\theta) &= \frac{k}{2\pi} \int d^2 z (\partial r \bar{\partial} r + \tanh^2 r \partial\theta\bar{\partial}\theta) \\ &= \frac{k}{4\pi} \int d^2 x \sqrt{h} h^{ij} (\partial_i r \partial_j r + \tanh^2 r \partial_i \theta \partial_j \theta) - \frac{1}{8\pi} \int d^2 x \sqrt{h} \Phi(r,\theta) R. \end{aligned} \tag{3.56}$$

The expression for the dilaton can be obtained demanding that the β-function equation to one loop order*

* Note that this equation has a different normalization w.r.t. (2.14) and that the tachyon is supposed to be small, so that contributions of $O(T^2)$ are neglected.

$$R_{ab} = D_a D_b \Phi \tag{3.57}$$

should be satisfied. The result is

$$\Phi = 2 \log \cosh r + \mathrm{const.} \tag{3.58}$$

The antisymmetric tensor field $B_{\mu\nu}$ can be gauged away in a $(1+1)$-dimensional target space so that it has not to be taken into account. The condition that one is considering small tachyons gives an equation of motion for this field that allows us to determine its form as a function the coordinates [5]. We will do this in section 3.5.

If we compare the obtained action with Liouville theory (2.19) coupled to $c = 1$ matter, we see that for large r this field can be identified with the Liouville field ϕ. The precise relation between the two theories will be one important point that we will later explore in more detail.

To obtain the space-time interpretation as a 2D black hole we will make the analytic continuation to Lorentz signature. Naively, the Minkowski black hole can be obtained with the redefinition $\theta = it$, so that the line element has the form

$$ds^2 = dr^2 - \tanh^2 r dt^2. \tag{3.59}$$

The above metric has a singularity at $r = 0$ that turns out to be only a coordinate singularity, since the scalar curvature

$$\mathcal{R} = \frac{4}{\cosh^2 r}$$

is regular at this point. In order to get a parametrization of the complete space-time, including the regions past the singularity, we will make the coordinate transformations

$$2v = e^{r'+t}, \qquad 2u = -e^{r'-t}, \qquad \text{where} \qquad r' = r + \ln(1 - e^{-2r}). \tag{3.60}$$

The line element then has the form

$$ds^2 = -\frac{dudv}{1 - uv}. \tag{3.61}$$

This metric exhibits all the space-time regions from the ordinary Schwarzschild solution (2.6), with the horizons at $uv = 0$ as well as a curvature singularity at $uv = 1$.

Instead of considering the formal analytic continuation $\theta = it$, the Minkowski version of the black hole can also be obtained gauging a different subgroup of SL(2,$\mathbb{R}$). In this case we consider the noncompact one parameter group generated by

$$\delta g = \varepsilon \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g + g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]. \tag{3.62}$$

We can parametrize the SL(2,$\mathbb{R}$) group element as

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}, \qquad \text{with} \qquad ab + uv = 1. \tag{3.63}$$

This is a global parametrization. The SL(2,$\mathbb{R}$) coset is a double cover of the (u, v) plane, depending on the sign of a and b. These variables can be eliminated with a particular choice of gauge. In the region $1 - uv > 0$, one can fix the gauge $a = b$

because either $a, b > 0$ or $a, b < 0$ holds. In the region $1 - uv < 0$, we can choose the gauge $a = -b$. In both cases we obtain with (3.7) and (3.63) the action

$$\mathcal{S} = -\frac{k}{4\pi} \int d^2x \sqrt{h} \frac{h^{ij} \partial_i u \partial_j v}{1 - uv}. \tag{3.64}$$

The corresponding line element is represented by (3.61). This action describes the regions I-VI of the Lorentzian black hole.

3.5 The Exact Background of Dijkgraaf, Verlinde and Verlinde

The target space geometry of the quantum theory for finite k was found by Dijkgraaf, H. Verlinde and E. Verlinde [7]. In their approach the gauge field A is not integrated out, so that corrections in $1/k$ can be taken into account. The form of the exact metric follows by comparing the form of the Klein-Gordon operator (2.22) with the L_0-operator that follows from the group theory of SL(2,$\mathbb{R}$), as we will now see. The action of the SL(2,$\mathbb{R}$)-gauged WZW model can be written by parametrizing g in terms of Euler angles:

$$g = e^{i\theta_L \sigma_2/2} e^{r\sigma_1/2} e^{i\theta_R \sigma_2/2}. \tag{3.65}$$

Here σ_i are Pauli matrices and the ranges of the fields are $0 \leq r \leq \infty$, $0 \leq \theta_L < 2\pi$ and $-2\pi \leq \theta_R < 2\pi$. This is a suitable parametrization for the Euclidean black hole, while for the Minkowski case we have to use

$$g = e^{it_L \sigma_3/2} e^{r\sigma_1/2} e^{it_R \sigma_3/2}, \tag{3.66}$$

where $0 \leq r \leq \infty$, $-\infty \leq t_{R,L} < \infty$. After introducing the form of g, the action (3.9) for the Euclidean theory can be written in the form:

$$S^{gf}_{WZW} = \frac{k}{4\pi}\int d^2z(\bar{\partial} r\partial r - \bar{\partial}\theta_L\partial\theta_L - \bar{\partial}\theta_R\partial\theta_R - 2\cosh r\bar{\partial}\theta_L\partial\theta_R) + S(X) + S(B,C). \tag{3.67}$$

The Lorentz gauge condition for A has been imposed, so that the fields X and (B,C) appear as previously explained. We can calculate the form of the conserved currents (3.12) in terms of Euler angles. The expressions are given by

$$\begin{aligned} J^3(z) &= k(\partial\theta_L + \cosh r\partial\theta_R) \\ J^{\pm}(z) &= ke^{\pm i\theta_L}(\partial r \pm i\sinh r\partial\theta_R). \end{aligned} \tag{3.68}$$

The primary fields of the ungauged theory can be represented using the matrix elements of the different representations of SL(2,$\mathbb{R}$) [49]:

$$T(r,\theta_R,\theta_L) = \langle j,\omega_L|g(r,\theta_L,\theta_R)|j,\omega_R\rangle. \tag{3.69}$$

The quantum numbers ω_L and ω_R are the eigenvalues of the currents J^3_0 and $\bar{J}^3_0$ respectively. The form of the primary fields of the coset theory is determined by (3.45). The invariance under $Q^{\mathrm{U}(1)}$ implies the relations

$$\omega_L + m = 0 \qquad \text{and} \qquad \omega_R - \bar{m} = 0. \tag{3.70}$$

We can shift the fields $\theta_L \to \theta_L - X$ and $\theta_R \to \theta_R + \bar{X}$ so that the dependence on the field X disappears from the primary fields. For the Euclidean theory primary fields can be represented through the Jacobi functions

$$T(r,\theta_L,\theta_R) = \mathcal{P}^j_{\omega_L\,\omega_R}(\cosh r)e^{i\omega_L\theta_L + i\omega_R\theta_R}. \tag{3.71}$$

We will now restrict ourselves to the mini-superspace description of the problem. This means that we keep only the zero-mode algebra:

$$\begin{aligned}
&[\mathcal{J}_3, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm \\
&[\mathcal{J}_+, \mathcal{J}_-] = -2\mathcal{J}_3.
\end{aligned} \tag{3.72}$$

These zero modes are represented as differential operators in the following way:

$$\begin{aligned}
&\mathcal{J}_3 = -i\frac{\partial}{\partial\theta_L}, \\
&\mathcal{J}_\pm = e^{\pm i\theta_L}\left(\frac{\partial}{\partial r} \mp \frac{i}{\sinh r}\left(\frac{\partial}{\partial\theta_R} - \cosh r\frac{\partial}{\partial\theta_L}\right)\right).
\end{aligned} \tag{3.73}$$

In this representation the zero-mode Casimir of SL(2,$\mathbb{R}$) that follows from the Sugawara prescription takes the form:

$$\Delta_0 = \frac{\partial^2}{\partial r^2} + \coth r\frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\left(\frac{\partial^2}{\partial\theta_L^2} - 2\cosh r\frac{\partial^2}{\partial\theta_L\partial\theta_R} + \frac{\partial^2}{\partial\theta_R^2}\right). \tag{3.74}$$

The complete L_0 and $\bar{L}_0$ operators follow, taking into account the free boson X. Their expressions are

$$L_0 = -\frac{\Delta_0}{k-2} - \frac{1}{k}\frac{\partial^2}{\partial\theta_L^2},$$
$$\bar{L}_0 = -\frac{\Delta_0}{k-2} - \frac{1}{k}\frac{\partial^2}{\partial\theta_R^2}. \tag{3.75}$$

The on-shell conditions (3.44) imply that a physical state is annihilated by the operator:

$$L_0 - \bar{L}_0 = \frac{1}{k}\left(\frac{\partial^2}{\partial\theta_R^2} - \frac{\partial^2}{\partial\theta_L^2}\right). \tag{3.76}$$

Therefore, these states can be split in the following way:

$$T(r,\theta_L,\theta_R) = T(r,\theta) + \widetilde{T}(r,\widetilde{\theta}), \tag{3.77}$$

where

$$\theta = \frac{1}{2}(\theta_L + \theta_R) \qquad\qquad \widetilde{\theta} = \frac{1}{2}(\theta_L - \theta_R). \tag{3.78}$$

States that depend only on θ are the momentum modes, while the winding modes depend only on $\widetilde{\theta}$. The L_0 operator has a different form depending on whether it acts on $T(r,\theta)$ or $\widetilde{T}(r,\widetilde{\theta})$. Therefore both operators will lead to different target spaces that correspond to dual manifolds. When the operator L_0 acts on $T(r,\theta)$ it takes the form

$$L_0 = -\frac{1}{k-2}\left(\frac{\partial^2}{\partial r^2} + \coth r\frac{\partial}{\partial r} + \left(\coth^2\frac{r}{2} - \frac{2}{k}\right)\frac{\partial^2}{\partial\theta^2}\right). \tag{3.79}$$

The expressions for the metric and the dilaton for finite k follow from the comparison with (2.22) and take the form:

$$ds^2 = \frac{k-2}{2}\left(dr^2 + \beta^2(r)d\theta^2\right) \qquad \text{and} \qquad \Phi = \log\left(\frac{\sinh r}{\beta(r)}\right), \tag{3.80}$$

where $\beta(r)$ is given by

$$\beta(r) = 2\left(\coth^2\frac{r}{2} - \frac{2}{k}\right)^{-\frac{1}{2}}. \tag{3.81}$$

To leading order in $1/k$ this coincides with the semiclassical metric, found by Witten (3.55) after a simple rescaling of the coordinate r. The space- time diagram in this limit is the semi-infinite cigar represented in Fig. 2. It has been verified that these expressions are the perturbative solution of the β-function equations up to three [8] and four loops [9]. When it acts on $\widetilde{T}(r,\widetilde{\theta})$ the operator has the form:

$$L_0 = -\frac{1}{k-2}\left(\frac{\partial^2}{\partial r^2} + \coth r\frac{\partial}{\partial r} + \left(\tanh^2\frac{r}{2} - \frac{2}{k}\right)\frac{\partial^2}{\partial\widetilde{\theta}^2}\right). \tag{3.82}$$

It leads to a background metric and dilaton (3.80) with

$$\beta(r) = 2\left(\tanh^2\frac{r}{2} - \frac{2}{k}\right)^{-\frac{1}{2}}. \tag{3.83}$$

This metric and the corresponding dilaton have a singularity at $r = \text{arc tanh}\sqrt{2/k}$. In the semiclassical limit $k \to \infty$ the line element takes the form

$$ds^2 = dr^2 + 4\coth^2\frac{r}{2}d\widetilde{\theta}^2, \tag{3.84}$$

so that the winding modes propagate on a manifold that looks like a "trumpet" (Fig. 3). We have a real singularity at $r = 0$.

Fig. 3: Target space geometry of the Euclidean black hole on which winding modes propagate in the limit $k \to \infty$.

The metric that describes the propagation of winding modes (3.84) can be obtained from the one describing the propagation of momentum modes (3.80) to all orders in $1/k$ by the transformation

$$r \to r + i\pi \qquad \text{and} \qquad \theta \to \widetilde{\theta}, \tag{3.85}$$

or equivalently $r \to r + i\pi/2$ in the notation (3.55).

In the last section we have seen that if we choose a suitable parametrization of the SL(2,$\mathbb{R}$)-valued field g in terms of the coordinates (u, v) and gauging a noncompact subgroup of this symmetry group, we are able to obtain all the regions of the maximally extended Minkowskian version of the Schwarzschild black hole. Instead of formulating the maximal extension of the Minkowski black hole in (u, v) coordinates, we can use the (r, t) coordinates by choosing a suitable range of the fields. By introducing the variables

$$r = \log\left(\sqrt{1 - uv} + \sqrt{-uv}\right) \qquad \text{and} \qquad t = \frac{1}{2}\log\left(-\frac{u}{v}\right), \tag{3.86}$$

where r is real or imaginary depending on which region of the Schwarzschild black hole we want to describe, this metric can also be written in the form

$$ds^2 = dr^2 - \tanh^2 r\, dt^2. \tag{3.87}$$

r takes the values:

$$r \in \begin{cases} [0,\infty] & \text{in region I where } uv \leq 0 \\ i[0,\pi/2] & \text{in region III where } 0 \leq uv < 1 \\ [0,\infty] + i\pi/2 & \text{in region V where } 1 \geq uv \end{cases} \tag{3.88}$$

The metric (3.87) can then be written the following form [7]

$$ds^2 = \begin{cases} dr^2 - \tanh^2 r dt^2 & \text{in region I with } r \in \{0,\infty\} \\ -dr^2 + \tan^2 r dt^2 & \text{in region III with } r \in \{0,\pi/2\} \\ dr^2 - \coth^2 r dt^2 & \text{in region V with } r \in \{0,\infty\} \end{cases} \tag{3.89}$$

where we have redefined $r \to r/i$ in region III and $r \to r - i\pi/2$ in region V. We can now make a correspondence with the different regions of the Euclidean black hole. We observe that regions I and V of the Minkowski black hole correspond to the cigar and the trumpet in the Euclidean theory. The metric of region III can be seen as the analytic continuation of a metric of the form:

$$ds^2 = dr^2 + \tan^2 r d\theta^2 \tag{3.90}$$

The manifold that corresponds to this metric has the shape of a "hat"⋆ as shown in Fig. 4. As a CFT it can be regarded as the SU(2)/U(1) coset, that is a parafermionic model. [7,52]. This model can be shown to be self-dual. The form of the Euclidean metric in regions I, III and V is represented in Fig. 4.

⋆ I thank H. Verlinde for pointing this out to me.

Fig. 4: Penrose diagram of the maximally extended Minkowski black hole and the corresponding regions in the Euclidean black hole. The arrows indicate the time flow $\partial_t = u\partial_u + v\partial_v$.

4. FREE FIELD APPROACH TO THE BLACK HOLE CFT

In the mini-superspace description of the black hole CFT, we have considered field configurations that are independent of the space coordinate and we kept only the zero mode of the fields. If we would like to study the full string theory, we have to find a suitable formulation of the problem that allows us to actually make computations and, at the same time, to keep all the modes of the fields. Such a formulation is the representation of the CFT in terms of free-fields. This representation has already played an important role in the computation of correlation functions of minimal models or the amplitudes of gravitationally dressed fields in non-critical string theory.

The WZW model can be written in terms of free fields using the Gauss-decomposition [28,29]. Equivalently, one can write the SL(2,$\mathbb{R}$) current algebra in terms of the Wakimoto representation [53]. For the SU(2) CFT, Dotsenko [54] has used this representation to compute certain correlation functions of primary fields for spherical topologies. The obtained correlators agree with those computed by Fateev and A. B. Zamolodchikov [55] using the SU(2) Ward identities. In the case of the black hole CFT, Bershadsky and Kutasov [27] have proposed to use the Wakimoto representation of SL(2,$\mathbb{R}$) to compute interesting quantities for the black hole, as for example the $\mathcal{S}$-matrix of tachyons interacting in the black hole background [32]. We will now present the free-field approach to the SL(2,$\mathbb{R}$)/U(1) coset model.

4.1 The Operator Approach

The SL(2,$\mathbb{R}$) Wakimoto representation * is formulated in terms of three free-fields β, γ and ϕ . The chiral bosonic superconformal ghosts β-γ have spin 1 and 0, respectively, and they are described by the action

$$S = \frac{1}{2\pi} \int d^2 z (\beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma}). \tag{4.1}$$

The two-point function of these fields is

$$\langle \gamma(z) \beta(w) \rangle = -\frac{1}{(z-w)},$$

and the same for the antiholomorphic fields $\bar{\gamma}$ and $\bar{\beta}$. The other OPEs are regular and we have no contractions between holomorphic and antiholomorphic ghosts. To perform concrete calculations it will be useful to bosonize the β-γ system as follows [56]:

$\star$ In the following we will use the conventions of ref. [24]

$$\beta = -i\partial v e^{iv-u}, \qquad \gamma = e^{u-iv}, \tag{4.2}$$

where u and v are ordinary bosons

$$\langle u(z)u(w)\rangle = \langle v(z)v(w)\rangle = -\log(z-w). \tag{4.3}$$

From this bosonization prescription we see that we are able to define non-integer powers of the operator γ, while we can only define positive integer powers of β since this operator contains derivatives. The field ϕ is an ordinary non-compact free boson with a background charge

$$S = \frac{1}{2\pi}\int \frac{1}{2}\partial\phi\bar{\partial}\phi - \frac{2}{\alpha_+}R^{(2)}\phi \tag{4.4}$$

and the propagator

$$\langle \partial\phi(z)\partial\phi(w)\rangle = -\frac{1}{(z-w)^2}.$$

We have introduced the notation $\alpha_+^2 = 2k-4$. The currents that satisfy the OPE (3.13) have the following form:

$$J_+(z) = \beta(z)$$

$$J_3(z) = -\beta(z)\gamma(z) - \frac{\alpha_+}{2}\partial\phi(z) \tag{4.5}$$

$$J_-(z) = \beta(z)\gamma^2(z) + \alpha_+\gamma(z)\partial\phi(z) + k\partial\gamma(z).$$

This is easy to check, using the previous two-point functions. The energy-momentum tensor follows from the Sugawara prescription. After inserting the above form of the currents, we obtain

$$T_{\mathrm{SL}(2,\mathbb{R})} = \beta\partial\gamma - \frac{1}{2}(\partial\phi)^2 - \frac{1}{\alpha_+}\partial^2\phi. \tag{4.6}$$

It has a central charge as a function of the level k of the Kac-Moody algebra:

$$c = \frac{3k}{k-2}. \tag{4.7}$$

The complete action associated with the energy-momentum tensor (4.6) is therefore:

$$S = \frac{1}{2\pi}\int \frac{1}{2}\partial\phi\bar{\partial}\phi - \frac{2}{\alpha_+}R^{(2)}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}. \tag{4.8}$$

As we mentioned in section 3.2, the basic fields are the Kac-Moody primaries $|j, m\rangle$. In the Wakimoto representation they are created by the action of the following (not normalized) "tachyon" vertex operator on the SL(2) vacuum:

$$T_{j\ m}(z) =: \gamma^{j-m}(z)e^{\frac{2}{\alpha_+}j\phi(z)} :, \tag{4.9}$$

and the same expression for the antiholomorphic part. Using the free-field representation of the currents (4.5), it is easy to check that $T_{j\ m}$ satisfies the definition of a Kac-Moody primary (3.15), as well as (3.16) and (3.17). The conformal dimension of $T_{j\ m}$ is given by (3.29).

If we would like to calculate correlation functions of the vertex operators $T_{j\ m}$ we would need a screening charge, in order to guarantee the charge conservation arising from the zero mode integration. This screening operator can be determined from the observation that the ungauged model has the SL(2,$\mathbb{R}$) symmetry, so that the correlators have to satisfy the Ward Identities. Since these identities should be satisfied in the free-field representation, the screening charge must have a regular

OPE with the stress tensor and the currents. We must also take into account, the fact that only positive integer powers of β are well defined through bosonization. The screening that satisfies these conditions can be represented as the following surface integral:

$$\mathcal{Q} = \int d^2 z J(z,\bar{z}), \qquad J(z,\bar{z}) = \beta(z)\bar{\beta}(\bar{z}) e^{-\frac{2}{\alpha_+}\phi(z,\bar{z})}. \tag{4.10}$$

The operator $J(z,\bar{z})$ is no longer a Kac-Moody primary. It is one of the simplest operators at higher mass level, as we will discuss in more detail in the next chapter. One can easily check the identities [54]

$$\begin{aligned} J_3(z)J(w,\bar{w}) &\sim \text{reg.}, \\ J_+(z)J(w,\bar{w}) &\sim \text{reg.}, \\ J_-(z)J(w,\bar{w}) &\sim \frac{\partial}{\partial w}\left(\frac{e^{-\frac{2}{\alpha_+}\phi(w,\bar{w})}}{(z-w)}\right). \end{aligned} \tag{4.11}$$

The total derivative appearing in the last OPE requires a careful treatment of contact terms, as we will see when we calculate correlation functions in chapter 6. This comes from the fact that we are working with a surface integral and not with a contour integral, where this contribution generically vanishes. As is known to be correct for a Coulomb gas model or Liouville theory [57,58,34,59], the screening charge has to be added to the action and considered as the interaction of the model [27]. We will see in section 4.3 that this prescription indeed reproduces the correct values for the metric and the dilaton for finite k found in ref. [7]. The complete action of the ungauged model is therefore

$$S = \frac{1}{2\pi}\int \frac{1}{2}\partial\phi\bar{\partial}\phi - \frac{2}{\alpha_+}R^{(2)}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + 2\pi M\beta\bar{\beta}e^{-\frac{2}{\alpha_+}\phi}. \tag{4.12}$$

In the above expression there appears a free parameter M that is related to the black hole mass [27,24]. This becomes clear from the space-time interpretation of this model already at the semiclassical level.

As we previously saw, in order to construct the conformal field theory of the Euclidean black hole we are interested in the coset SL(2,$\mathbb{R}$)/U(1). This gauging must be done as explained in section 3.2. To gauge the U(1) subgroup, we introduce a gauge boson X and a pair of fermionic ghosts B, C of spin $1, 0$ respectively. The complete energy-momentum tensor of the U(1) gauged theory is therefore

$$T_{\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)} = \beta\partial\gamma - \frac{1}{2}(\partial\phi)^2 - \frac{1}{\alpha_+}\partial^2\phi - \frac{1}{2}(\partial X)^2 - B\partial C. \tag{4.13}$$

Taking into account the fermionic diffeomorphism ghosts (b, c) the total central charge of the theory is zero. This means that the complete gauge-fixed action (without the fermionic ghosts) is*

$$S = \frac{1}{2\pi}\int \frac{1}{2}\partial\phi\bar{\partial}\phi + \frac{1}{2}\partial X\bar{\partial}X - \frac{2}{\alpha_+}R^{(2)}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + 2\pi M\beta\bar{\beta}e^{-\frac{2}{\alpha_+}\phi}. \tag{4.14}$$

For the Euclidean theory the gauge boson X is compact.

The two BRST charges are given by (3.39) and (3.40) with the previous representation for the currents and the energy-momentum tensor. The tachyon states of the gauged theory are the dressed ghost number zero primary fields, which are invariant under $Q^{U(1)\,\dagger}$

$$\mathcal{V}_{j\,m} =: \gamma^{j-m} e^{\frac{2}{\alpha_+}j\phi} e^{im\sqrt{\frac{2}{k}}X} : . \tag{4.15}$$

The characteristics of these fields have been mentioned in section 3.2.

$\star$ The fermionic ghost contribution can be generically factorized out of the correlation functions on the sphere.

$\dagger$ Since j and m are arbitrary at this moment, these operators will be called tachyons.

4.2 The Gauss Decomposition

Briefly, we would like to mention that the model introduced by Bershadsky and Kutasov can be obtained directly from the Lagrangian formulation of the WZW model, using the Gauss decomposition [28,29,33,60] ‡. The Wakimoto representation of the current algebra is equivalent to the choice of a particular representation for the SL(2,$\mathbb{R}$) valued field g. As we previously explained this representation is suitable to evaluate the full string theory. The SL(2,$\mathbb{R}$)-valued field g is parametrized with the Gauss decomposition as follows [29,60]

$$g = g(\gamma, \phi, \bar{\gamma}) = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} e^{\phi/2} & 0 \\ 0 & e^{-\phi/2} \end{pmatrix} \begin{pmatrix} 1 & \bar{\gamma} \\ 0 & 1 \end{pmatrix}. \tag{4.16}$$

Here all the fields are real and the boson ϕ is non-compact with a range $-\infty < \phi < \infty$. We can insert this expression for g into the ungauged WZW action (4.12). Using the Polyakov-Wiegmann formula [45]

$$S_{WZW}(GH^{-1}) = S_{WZW}(G) + S_{WZW}(H) + \frac{1}{16\pi} \int \mathrm{tr}\left(G^{-1}\bar{\partial}GH^{-1}\partial H\right) d^2\xi \tag{4.17}$$

the ungauged model can be written as

$$S_{WZW} = \frac{1}{16\pi} \int \left(e^{-\phi}\bar{\partial}\gamma\partial\bar{\gamma} + \partial\phi\bar{\partial}\phi\right). \tag{4.18}$$

If we introduce two auxiliary fields β and $\bar{\beta}$ the action becomes

‡ I thank E. Verlinde for providing me a copy of his notes.

$$S_{WZW} = \frac{1}{16\pi} \int \left(\partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}e^{-\phi} \right). \tag{4.19}$$

This is precisely the action of Bershadsky and Kutasov after a trivial rescaling of the fields and up to quantum corrections. Of course, we have to be careful with the insertion of equations of motion if we pass to the quantum Lagrangian. Then we have to keep Jacobians into account after making a change of variables [28]. This is, for example, the origin of the scalar curvature term in the action (4.12). The free parameter corresponding to the black hole mass M can be obtained from a shift in ϕ.

5. TARGET SPACE GEOMETRY IN TERMS OF WAKIMOTO COORDINATES

In this section we are going to show that the SL(2,$\mathbb{R}$)/U(1) gauged WZW model, formulated in terms of Wakimoto coordinates, has the same space-time interpretation for finite k as the one found by Dijkgraaf et al. [30]. We will derive the expressions for the metric and the dilaton for finite k of the 2D black hole in terms of free-field coordinates identifying the L_0 operator with the target space Laplacian of the σ-model* .

The mini-superspace approximation of the currents (4.5) takes the following form:

$$
\begin{aligned}
\mathcal{J}_+ &= \frac{\partial}{\partial\gamma} \\
\mathcal{J}_3 &= -\gamma\frac{\partial}{\partial\gamma} + \frac{1}{2}\frac{\partial}{\partial\phi} \\
\mathcal{J}_- &= \gamma^2\frac{\partial}{\partial\gamma} - \gamma\frac{\partial}{\partial\phi} - e^{-2\phi}\frac{\partial}{\partial\bar{\gamma}}.
\end{aligned}
\tag{5.1}
$$

We would like to remark that the zero mode algebra is also satisfied if we parametrize J_- as

$$
\mathcal{J}_- = \gamma^2\frac{\partial}{\partial\gamma} - \gamma\frac{\partial}{\partial\phi} + \epsilon e^{-2\phi}\frac{\partial}{\partial\bar{\gamma}},
\tag{5.2}
$$

where ϵ is an arbitrary free parameter, related to the black hole mass. We will discuss this possibility at the end and we will use (5.1) for the moment being.

$\star$ We would like to thank E. Verlinde for helping us to understand this space-time interpretation.

The expressions (5.1) can be computed following the steps that lead to the quantum-mechanical description of Liouville theory [61,62]. First, we have to use the canonical quantization procedure and map the complex plane $(z, \bar{z})$ to the cylinder (t, σ)

$$z = e^{t+i\sigma}, \qquad \bar{z} = e^{t-i\sigma}. \tag{5.3}$$

Then we expand the fields using a Fourier decomposition. For $\phi(\sigma, t)$ and its canonical conjugate momentum $\Pi(\sigma, t)$, this expansion takes the form

$$\begin{aligned}
\phi(\sigma, t) &= \phi_0(t) + \sum_{n\neq 0} \frac{i}{n}(a_n(t)e^{-in\sigma} + b_n(t)e^{in\sigma}) \\
\Pi(\sigma, t) &= p_0(t) + \sum_{n\neq 0} \frac{1}{4\pi}(a_n(t)e^{-in\sigma} + b_n(t)e^{in\sigma}).
\end{aligned} \tag{5.4}$$

Since from this point of view $\phi(\sigma, t)$ is not a free-field the time dependence of the components might be complicated.

In order to quantize the theory we impose the equal-time commutator

$$[\phi(\sigma, t), \Pi(\sigma', t)] = \delta(\sigma - \sigma'). \tag{5.5}$$

In the mini-superspace description we consider field configurations that are not dependent on σ and we keep only the zero modes of the fields $\phi_0(t)$ and $p_0(t)$. We replace the canonical-conjugate momenta by derivatives with respect to the fields†:

† We drop the subindex 0 of the zero modes.

$$
\begin{aligned}
\Pi_\phi &= \frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial \phi} \\
\Pi_\gamma &= \beta = \frac{\partial}{\partial \gamma}.
\end{aligned}
\tag{5.6}
$$

To obtain the complete expression for $\mathcal{J}_-$ we have to replace $\partial\gamma$ by an appropriate expression. Here we use the equation of motion for β and find:

$$
\frac{\partial \gamma}{\partial t} = -\bar{\beta} e^{-\frac{2}{\alpha_+}\phi} = -e^{-\frac{2}{\alpha_+}\phi} \frac{\partial}{\partial \bar{\gamma}}. \tag{5.7}
$$

With these substitutions, we obtain the expression (5.1) for the currents, after rescaling ϕ by a factor of α_+, $\phi \rightarrow \alpha_+\phi$. For the antiholomorphic currents we follow exactly the same procedure and obtain

$$
\begin{aligned}
\bar{\mathcal{J}}_+ &= \frac{\partial}{\partial \bar{\gamma}} \\
\bar{\mathcal{J}}_3 &= -\bar{\gamma}\frac{\partial}{\partial \bar{\gamma}} + \frac{1}{2}\frac{\partial}{\partial \phi} \\
\bar{\mathcal{J}}_- &= \bar{\gamma}^2 \frac{\partial}{\partial \bar{\gamma}} - \bar{\gamma}\frac{\partial}{\partial \phi} - e^{-2\phi}\frac{\partial}{\partial \gamma}.
\end{aligned}
\tag{5.8}
$$

We note that although $\mathcal{J}_-$ contains a $\bar{\gamma}$ dependence all the antiholomorphic currents $\bar{\mathcal{J}}_i$ commute with the holomorphic currents $\mathcal{J}_i$. Here we have chosen a particular normal-ordering prescription for the β-γ system; other normal-ordering prescriptions are equivalent to this one after a redefinition of the wave-functions.

Independently, we can construct these currents using some elements of group theory. The currents are the generators of the three one-parameter subgroups of SL(2,$\mathbb{R}$) represented by the following matrices [49]:

$$\omega_1(t) = g(t,0,0) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\omega_2(t) = g(0,t,0) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \tag{5.9}$$

$$\omega_3(t) = g(0,0,t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

The infinitesimal generators $\widehat{A}_i$ corresponding to these transformations are defined by

$$\widehat{A}_i = \left(\frac{\partial\gamma(t)}{\partial t}\frac{\partial}{\partial\gamma} + \frac{\partial\phi(t)}{\partial t}\frac{\partial}{\partial\phi} + \frac{\partial\bar{\gamma}(t)}{\partial t}\frac{\partial}{\partial\bar{\gamma}} \right)_{t=0}. \tag{5.10}$$

Here $\gamma(t)$, $\phi(t)$ and $\bar{\gamma}(t)$ are identified from the definition:

$$g(\gamma(t), \phi(t), \bar{\gamma}(t)) = \omega_i(t) \cdot g(\gamma, \phi, \bar{\gamma}). \tag{5.11}$$

The expressions that we obtain for the infinitesimal generators therefore are:

$$\widehat{A}_1 = e^{-2\phi}\frac{\partial}{\partial\bar{\gamma}} + \gamma\frac{\partial}{\partial\phi} - \gamma^2\frac{\partial}{\partial\gamma}$$

$$\widehat{A}_2 = \frac{\partial}{\partial\phi} - 2\gamma\frac{\partial}{\partial\gamma} \tag{5.12}$$

$$\widehat{A}_3 = \frac{\partial}{\partial\gamma}.$$

This means that, after identifying

$$\widehat{A}_1 = -\mathcal{J}_-, \qquad \widehat{A}_2 = 2\mathcal{J}_3, \qquad \widehat{A}_3 = \mathcal{J}_+, \tag{5.13}$$

we obtain the same result as we had from the mini-superspace approximation of the currents. The antiholomorphic generators are obtained in the same way. In this case we have to set

$$g(\gamma(t), \phi(t), \bar{\gamma}(t)) = g(\gamma, \phi, \bar{\gamma}) \cdot \omega_i(t), \tag{5.14}$$

which gives us the antiholomorphic currents (5.8).

With the mini-superspace approximations for the currents, we now obtain the expressions for the zero-mode Casimir using Sugawara's prescription (3.24)

$$\Delta_0 = \frac{\partial^2}{\partial \phi^2} + \frac{\partial}{\partial \phi} + e^{-\phi} \frac{\partial^2}{\partial \gamma \partial \bar{\gamma}}. \tag{5.15}$$

Here we have rescaled ϕ by a factor of 2, $\phi \to \phi/2$. This means that the complete Klein-Gordon operators L_0 and $\bar{L}_0$ are given by

$$\begin{aligned} L_0 &= -\frac{\Delta_0}{k-2} - \frac{1}{k} \frac{\partial^2}{\partial X^2} \\ \bar{L}_0 &= -\frac{\Delta_0}{k-2} - \frac{1}{k} \frac{\partial^2}{\partial \bar{X}^2}. \end{aligned} \tag{5.16}$$

Therefore we are left with five fields: $\phi, \gamma, \bar{\gamma}, X, \bar{X}$. If we take the BRST constraints into account we see that not all of these variables are independent. We will eliminate γ and $\bar{\gamma}$ using the BRST conditions and keep only the variables $\phi, X, \bar{X}$ to make the space-time interpretation. The diffeomorphism invariance means that the physical states are dimension $(1, 1)$ fields, so that they satisfy

$$(L_0 - \bar{L}_0)\Psi = \left(\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial \bar{X}^2} \right) \Psi = 4 \frac{\partial^2 \Psi}{\partial X^+ \partial X^-} = 0, \tag{5.17}$$

where we have introduced coordinates $X^{\pm} = X \pm \bar{X}$ and Ψ are the wave functions associated to the states. This means that we have two types of wave functions, one that depends only on X^+ and one that depends only on X^-:

$$\Psi^+ = \Psi^+(\gamma, \bar{\gamma}, \phi, X^+) \qquad \text{and} \qquad \Psi^- = \Psi^-(\gamma, \bar{\gamma}, \phi, X^-). \tag{5.18}$$

The total wave function is then the sum $\Psi = \Psi^+ + \Psi^-$. One of these wave functions will represent the winding modes and the other one the momentum modes. The U(1) constraint eliminates the γ, $\bar{\gamma}$ dependence of the L_0 and $\bar{L}_0$ operators, as we will now see. First we notice that for Ψ this constraint implies*

$$\begin{aligned} \mathcal{J}_3 \Psi &= \left(\frac{\partial}{\partial \phi} - \gamma \frac{\partial}{\partial \gamma} - i \frac{\partial}{\partial X} \right) \Psi = 0 \\ \bar{\mathcal{J}}_3 \Psi &= \left(\frac{\partial}{\partial \phi} - \bar{\gamma} \frac{\partial}{\partial \bar{\gamma}} - i \frac{\partial}{\partial \bar{X}} \right) \Psi = 0. \end{aligned} \tag{5.19}$$

When acting on Ψ^+ or Ψ^- we therefore obtain the conditions:

$$\begin{aligned} \mathcal{J}_3 \Psi^+ &= \left(\frac{\partial}{\partial \phi} - \gamma \frac{\partial}{\partial \gamma} - i \frac{\partial}{\partial X^+} \right) \Psi^+ = 0, \qquad \mathcal{J}_3 \Psi^- = \left(\frac{\partial}{\partial \phi} - \gamma \frac{\partial}{\partial \gamma} - i \frac{\partial}{\partial X^-} \right) \Psi^- = 0, \\ \bar{\mathcal{J}}_3 \Psi^+ &= \left(\frac{\partial}{\partial \phi} - \bar{\gamma} \frac{\partial}{\partial \bar{\gamma}} - i \frac{\partial}{\partial X^+} \right) \Psi^+ = 0, \qquad \bar{\mathcal{J}}_3 \Psi^- = \left(\frac{\partial}{\partial \phi} - \bar{\gamma} \frac{\partial}{\partial \bar{\gamma}} + i \frac{\partial}{\partial X^-} \right) \Psi^- = 0. \end{aligned} \tag{5.20}$$

This means that the γ and $\bar{\gamma}$ dependence of the wave functions Ψ^+ and Ψ^- can be

⋆ Here we have two choices for $\bar{\mathcal{J}}_3^{total} = \bar{\mathcal{J}}_3 \pm i\partial/\partial X$. We have taken the "-" sign. The two solutions are again related by a duality transformation.

eliminated and their dependence on the Wakimoto coordinates is:

$$\Psi^+ = \Psi^+\left(\gamma\bar{\gamma}e^{\frac{1}{2}(\phi+iX^+)}, e^{\frac{1}{2}(\phi-iX^+)}\right) = \Psi^+\left(e^{\frac{1}{2}(\widetilde{\phi}+i\widetilde{X}^+)}, e^{\frac{1}{2}(\widetilde{\phi}-i\widetilde{X}^+)}\right) \qquad (5.21)$$

and

$$\Psi^- = \Psi^-\left(\gamma e^{\frac{1}{2}(\phi-iX^-)}, \bar{\gamma}e^{\frac{1}{2}(\phi+iX^-)}\right) = \Psi^-\left(e^{\frac{1}{2}(\widetilde{\phi}-i\widetilde{X}^-)}, e^{\frac{1}{2}(\widetilde{\phi}+i\widetilde{X}^-)}\right), \qquad (5.22)$$

where we have defined new variables $\widetilde{\phi}$ and $\widetilde{X}^{\pm}$. In terms of these coordinates the γ and $\bar{\gamma}$ dependence of the of the L_0 and $\bar{L}_0$ operators has been eliminated and the SL(2,$\mathbb{R}$) zero-mode Casimir operators take the following form:

$$\Delta_0^+ = \frac{\partial^2}{\partial\widetilde{\phi}^2} + \frac{\partial}{\partial\widetilde{\phi}} + e^{-\widetilde{\phi}}\left(\frac{\partial}{\partial\widetilde{\phi}} - i\frac{\partial}{\partial\widetilde{X}^+}\right)^2, \qquad (5.23)$$

and

$$\Delta_0^- = \frac{\partial^2}{\partial\widetilde{\phi}^2} + \frac{\partial}{\partial\widetilde{\phi}} + e^{-\widetilde{\phi}}\left(\frac{\partial^2}{\partial\widetilde{\phi}^2} + \frac{\partial^2}{\partial\widetilde{X}^{-^2}}\right). \qquad (5.24)$$

The wave functions Ψ^- and Ψ^+ play a role analogous to that of the wave functions $T(r,\theta)$ and $\widetilde{T}(r,\widetilde{\theta})$, which describe the propagation of momentum and of winding modes respectively. This means that each of them will generate a different background corresponding to the cigar and the trumpet respectively. We introduce the variable r as

$$r = 2\ \mathrm{arcoth}\sqrt{1+e^{-\widetilde{\phi}}} \qquad \text{or} \qquad \widetilde{\phi} = 2\log\sinh\frac{r}{2}, \qquad (5.25)$$

and we observe that with the above redefinition the condition $-\infty < \phi < \infty$ holds since $0 < r < \infty$ is satisfied. We obtain, after the redefinition

$$\widetilde{\theta} = \widetilde{X}^{+} - i\log(e^{-\widetilde{\phi}} + 1), \tag{5.26}$$

the following form of the L_0 operator when acting on the momentum modes Ψ^{+}:

$$L_0\Psi^{+} = -\frac{1}{k-2}\left(\frac{\partial^2}{\partial r^2} + \coth r\frac{\partial}{\partial r} + \left(\tanh^2\frac{r}{2} - \frac{2}{k}\right)\frac{\partial^2}{\partial\widetilde{\theta}^2}\right)\Psi^{+}. \tag{5.27}$$

If this operator acts on the winding modes described by Ψ^{-}, we obtain

$$L_0\Psi^{-} = -\frac{1}{k-2}\left(\frac{\partial^2}{\partial r^2} + \coth r\frac{\partial}{\partial r} + \left(\coth^2\frac{r}{2} - \frac{2}{k}\right)\frac{\partial^2}{\partial\theta^2}\right)\Psi^{-}. \tag{5.28}$$

This is precisely the form of the L_0 operators of ref. [7]. We have introduced the notation $\widetilde{X}^{-} \equiv \theta$. This implies that we will have the same background interpretation, i.e. the same metric and dilaton (3.80) for finite k as was found in ref. [7]. The dilaton written in terms of $\widetilde{\phi}$ takes the form:

$$\Phi^{+} = \widetilde{\phi} + \log\sqrt{1 - \frac{2}{k}\left(1 + e^{-\widetilde{\phi}}\right)} \tag{5.29}$$

and

$$\Phi^{-} = \widetilde{\phi} + \log\sqrt{\left(e^{-\widetilde{\phi}} + 1\right)\left(e^{-\widetilde{\phi}} + 1 - 2/k\right)}. \tag{5.30}$$

From the above expression we observe that only in the limit $k \to \infty$ (and $\widetilde{\phi} \to \infty$ for the second expression) does the field $\widetilde{\phi}$ coincide with the dilaton. The first expression for the dilaton coincides with the one computed in [27] in the semiclassical limit.

We could now consider the possibility of changing the sign of the black hole mass, by choosing $\epsilon = -1$ in eq.(5.2). We then get the relation

$$\widetilde{\phi} = 2 \log \cosh \frac{r}{2} \tag{5.31}$$

In this case the role of $\widetilde{X}^+$ and $\widetilde{X}^-$ is interchange.

In this calculation the screening charge proposed in ref. [27] was an important ingredient (5.7). We conclude that this operator generates the correct background, even for finite k. This agreement with the background of ref. [7] is important since we are going to evaluate the correlation functions of this model for $k = 9/4$, so that a space-time interpretation for finite k is the one of relevance. It would be interesting to derive the above metric directly from the Lagrangian of ref. [27] with a similar computation as the one done in ref. [8].

We would like to make one final remark concerning the range of the fields for $\epsilon = -1$. We define $\phi^* = \frac{\alpha_+}{2} \log(Mk/2)$, that indicates the value of ϕ at the event horizon. The range of the field ϕ consists of two parts. The region $\phi^* \leq \phi < \infty$ describes the Euclidean version of region I (outside the horizon), while $-\infty < \phi \leq \phi^*$ is the Euclidean version of region III (between the singularity and the horizon).

6. COHOMOLOGY OF THE EUCLIDEAN BLACK HOLE

As a first step to see if there exist similarities between the black hole CFT and standard non-critical string theory, we would like to analyse the cohomologies of the two theories.

It is well known that the critical bosonic string in two dimensions has in its spectrum, apart from the tachyon, an infinite number of states at higher mass levels and discrete values of the momenta. They are the so-called discrete states [63] that play an important role in the black hole CFT, as we will later see.

6.1 The Classification of Distler and Nelson

Distler and Nelson [22] have used the SL(2,$\mathbb{R}$) representation theory and the standard coset construction to calculate all the physical states that could (in principle) occur in the black hole CFT. It is possible that the true spectrum is a subset of the spectrum that is allowed by representation theory. A definitive statement on the states that occur can be achieved through the computation of the scattering amplitudes. This will be done in chapter 6.

Since for the quantization of the black hole theory we have two BRST charges to characterize the physical states of the coset theory, one could in principle consider either the cohomology of Q^{total} or the iterated cohomology in which the states are annihilated by each of the charges instead of by their sum.

In ref. [22] it is claimed that the two cohomologies agree. In the spectrum of the Euclidean black hole there appear discrete states besides the tachyon. This is familiar from the spectrum of $c = 1$ matter coupled to Liouville theory. These fields are no longer Kac-Moody primaries. The (on-shell) tachyon vertex operators for $c = 1$ have the form

$$\exp\left(ip_X X\right)\exp(p_\phi \phi), \tag{6.1}$$

Fig. 5: Physical states of the Euclidean black hole: open circle $\bigcirc$ $\mathcal{C}$, filled circle $\bullet$ $\mathcal{D}$, dot $\cdot$ $\widetilde{\mathcal{D}}$, circled dot $\odot$ (double occupied). The special states satisfying $|m| = 3(j+1/2)$, that lay on the broken line are "discrete tachyons" that occur at zero mass level.

where

$$\pm p_X = p_\phi + \sqrt{2}, \tag{6.2}$$

is fixed by the on-shell condition. Comparing this with the on-shell condition of the coset model for $k = 9/4$, we can identify

$$p_\phi = 2\sqrt{2}j \qquad \text{and} \qquad p_X = \frac{2}{3}\sqrt{2}m. \tag{6.3}$$

We can compare the quantum numbers of the states that occur in $c = 1$ coupled to Liouville theory with the states that may occur in the Euclidean black hole. The conclusion in ref. [22] is that the massless spectrum corresponding to the tachyons is identical in $c = 1$ and in the black hole CFT. However, at higher mass levels there seem to appear differences between the two theories. In addition to the $c = 1$ discrete states, they obtained new discrete states which have no counterpart in $c = 1$. The discrete states classified in ref. [22] are:

$$
\begin{array}{llll}
\widetilde{\mathcal{D}}^{\pm}: & p_X = \pm\dfrac{2s-4r-1}{2\sqrt{2}}, & p_\phi = \dfrac{2s+4r-5}{2\sqrt{2}}, & \mathcal{N} = r(2s-1) \\[2ex]
\mathcal{D}^{\mp}: & p_X = \pm\dfrac{s-2r+1}{\sqrt{2}}, & p_\phi = \dfrac{s+2r-3}{\sqrt{2}}, & \mathcal{N} = s(2r-1) \\[2ex]
\mathcal{C}: & p_X = \dfrac{2(s-r)}{\sqrt{2}}, & p_\phi = \dfrac{2(s+r-1)}{\sqrt{2}}, & \mathcal{N} = 4sr,
\end{array}
\tag{6.4}
$$

where r and s are positive integers and $\mathcal{N}$ is the mass level.

The states $\widetilde{\mathcal{D}}^{\pm}$ are the new discrete states, while $\mathcal{D}^{\mp}$ appear in $c=1$ matter coupled to Liouville theory, as well as $\mathcal{C}$, and belong to the discrete and supplementary series of SL(2,$\mathbb{R}$), respectively. We will later see how part of this spectrum appears as poles in tachyon correlation functions, as it is known to hold for $c=1$ coupled to gravity [34,58,59].

Fig. 6: Physical states of the $c=1$ non-critical string.

6.2 The Simplest Discrete States

We have already seen how the tachyon operators, i.e. the Kac-Moody primaries, look like in the free-field representation. In this section we are going to see how the vertex operators of the simplest discrete states that occur in the Euclidean black hole can be represented in this approach. These states are created by acting with negative modes of the currents on the Kac-Moody primaries and are therefore no longer annihilated by the positive modes of them. They satisfy the on-shell condition

$$-\frac{j(j+1)}{k-2}+\frac{m^2}{k}+\mathcal{N}=1. \tag{6.5}$$

The form of the simplest discrete states has been computed in ref. [27,24] and they have the form

$$V_{\mathcal{N}}=\beta^{\mathcal{N}}\exp\left(\frac{2j}{\alpha_+}\phi+im\sqrt{\frac{2}{k}}X\right). \tag{6.6}$$

The invariance under $Q^{\mathrm{U}(1)}$ implies $m=\mathcal{N}+j$. If we now consider states of dimension $(1,1)$, this implies that for $k=9/4$ we get two series of solutions:

$$\begin{aligned} m&=\frac{3}{2}\mathcal{N}-\frac{3}{8} \qquad & j&=-\frac{3}{8}+\frac{\mathcal{N}}{2},\\ m&=\frac{3}{4}\mathcal{N}-\frac{3}{4} \qquad & j&=-\frac{3}{4}-\frac{\mathcal{N}}{4}. \end{aligned} \tag{6.7}$$

The previous states are special cases of the general classification of Distler and Nelson [22]. In particular we see that the first series in eq. (6.7) realizes some of the new discrete states. The second series describes discrete states that appear in $c=1$ that are on the wrong branch, i.e. that satisfy $j<-1/2$.

We now consider the first examples to see explicitly if there is a correspondence to discrete states of $c = 1$. The case $\mathcal{N} = 1$ of the first series corresponds to the state $(j, m) = (1/8, 9/8)$. After simple computations one can show that this state can be represented as

$$
\begin{aligned}
&V\,(j = 1/8, m = 9/8) = \\
&\gamma^{-1} \frac{\partial}{\partial z} \exp\left(\frac{\phi}{2\sqrt{2}} + i\frac{3}{2\sqrt{2}}X\right) + \left\{Q^{\mathrm{U}(1)}, b\gamma^{-1} \exp\left(-\frac{\phi}{2\sqrt{2}} + i\frac{3}{2\sqrt{2}}X\right)\right\} = \\
&\gamma^{-1}\left\{Q^{Diff}, \left[b_{-1}, \exp\left(\frac{\phi}{2\sqrt{2}} + i\frac{3}{2\sqrt{2}}X\right)\right]\right\} + \left\{Q^{\mathrm{U}(1)}, b\gamma^{-1} \exp\left(-\frac{\phi}{2\sqrt{2}} + i\frac{3}{2\sqrt{2}}X\right)\right\}.
\end{aligned}
\tag{6.8}
$$

Here $\{,\}$ and $[,]$ means the commutator and anti-commutator respectively. We see that this new discrete state of Distler and Nelson becomes BRST-trivial in the Wakimoto representation, since we are left with a total derivative that decouples from the amplitudes. It seems reasonable that the same holds for the rest of the new discrete states. That this is indeed the case will be checked through the computation of the scattering amplitudes. We will see that these states do not appear as poles in the amplitudes. This is in agreement with the result of the analysis of the free field cohomology carried out by Eguchi et al. [24], which we will present in the next section.

We now discuss the case $\mathcal{N} = 1$ of the second series that corresponds to $(j, m) = (-1, 0)$ and represents the screening charge. This operator was identified as the black hole mass operator in ref. [27,24], where the semiclassical interpretation of the background of the free-field model has been found. In this computation one does not use the "L_0-approach". Instead of this, one identifies the action of the free-field model (4.14) with the σ-model action (2.12). For this purpose one has to transform the action formulated in terms of Wakimoto coordinates to a σ-model form. This is equivalent to eliminating the β-γ system of this description and to use only the coordinates of $c = 1$. Of course, if we are working directly at the level of the action, i.e. in the full field theory description, we have to be careful with the transformation of variables. However, in the case of the representation in terms of Euler angles, Tseytlin [64] showed that a careful analysis of the quantum effective action leads to the correct metric for finite k. We will now restrict to the semiclassical approach presented in ref. [27,24]. We start eliminating the β contribution of the screening charge. This can be done using the $Q^{\mathrm{U}(1)}$ charge. Using the bosonization formula we have

$$V = \beta e^{-\frac{2}{\alpha_+}\phi} = -i\partial v e^{iv-u-\frac{2}{\alpha_+}\phi}. \tag{6.9}$$

After taking the antiholomorphic part into account, it is easy to see that the interaction can be written as

$$V \simeq e^{-u+iv}e^{-\bar{u}+i\bar{v}}\left(\frac{\alpha_+}{2}\partial\phi + i\sqrt{\frac{k}{2}}\partial X\right)\left(\frac{\alpha_+}{2}\bar{\partial}\phi + i\sqrt{\frac{k}{2}}\bar{\partial}X\right)\exp\left(-\frac{2}{\alpha_+}\phi\right), \tag{6.10}$$

where we have neglected a BRST-trivial part and we have chosen a "-" sign in eqn. (3.41). The exponential factors involve the u, v dependence only through the combination $u - v$. In correlation functions every contraction of u will be cancelled by the contraction of v. This means that the interaction term can be written as

$$V \simeq \left(\frac{\alpha_+}{2}\partial\phi + i\sqrt{\frac{k}{2}}\partial X\right)\left(\frac{\alpha_+}{2}\bar{\partial}\phi + i\sqrt{\frac{k}{2}}\bar{\partial}X\right)\exp\left(-\frac{2}{\alpha_+}\phi\right). \tag{6.11}$$

We find that the interaction, for $k = 9/4$, is the discrete state

$$W^{-}_{1,0}\bar{W}^{-}_{1,0} = \partial X \bar{\partial} X e^{-2\sqrt{2}\phi} \tag{6.12}$$

of $c = 1$, up to a total derivative. Therefore the action of the SL(2,$\mathbb{R}$)/U(1) Euclidean black hole is identical to the 2D non-critical string, with a different interaction. Instead of the cosmological constant that is a tachyon operator, the interaction is a discrete state

$$\mathcal{S} = \frac{1}{8\pi}\int \sqrt{\widehat{g}}\widehat{g}^{ab}(\partial_a\phi\partial_b\phi + \partial_a X\partial_b X) - \frac{1}{4\pi\alpha_+}\int \sqrt{\widehat{g}}\widehat{R}^{(2)}\phi + M\int V. \tag{6.13}$$

The parameter M is free and it can be eliminated with a shift in ϕ. The metric that follows from the above σ-model is [27]

$$\begin{aligned} G_{\phi\phi} &= 1 + M\frac{\alpha_+^2}{4}\exp\left(-\frac{2}{\alpha_+}\phi\right) \\ G_{\phi X} &= iM\frac{\alpha_+}{2}\sqrt{\frac{k}{2}}\exp\left(-\frac{2}{\alpha_+}\phi\right) \\ G_{XX} &= 1 - M\frac{k}{2}\exp\left(-\frac{2}{\alpha_+}\phi\right). \end{aligned} \tag{6.14}$$

We observe that this metric is not diagonal. To diagonalize it we introduce a new coordinate*:

$$\vartheta = X + if(\phi), \qquad \text{with} \qquad f'(\phi) = \frac{1}{e^{2\phi} - 1}. \tag{6.15}$$

The parameter M can be eliminated with a shift in ϕ. The resulting action takes the form:

$\star$ Here we have rescaled the fields ϕ and X. The presence of the i in the above metric can be avoided with a change to Minkowski conventions $X \to iX$ [24].

$$\mathcal{S} = \frac{k}{4\pi} \int \sqrt{\widehat{g}} \widehat{g}^{ab} \left(\frac{\partial_a \phi \partial_b \phi}{1 - e^{-2\phi}} + \left(1 - e^{-2\phi}\right) \partial_a \vartheta \partial_b \vartheta \right) - \frac{1}{4\pi} \int \sqrt{\widehat{g}} \widehat{R}^{(2)} \phi. \quad (6.16)$$

After introducing the variable r

$$\phi = \log \cosh r, \quad (6.17)$$

we are left with the SL(2,$\mathbb{R}$)/U(1) WZW action found by Witten (3.56) in the semiclassical limit. We can do the similar considerations for the dual σ-model in the semiclassical limit. We previously saw that this dual model can be obtained from (3.56) with the shift $r \rightarrow r + i\pi/2$. One arrives at the conclusion [27] that it corresponds to $M < 0$, and it describes region V (behind the singularity).

The scalar curvature is given by [24]

$$R = M \exp \left(-\sqrt{\frac{2}{k}} \phi \right), \quad (6.18)$$

so that the curvature singularity occurs for $\phi = -\infty$.

6.3 The Complete Free Field Cohomology

Eguchi et al. [24] have shown that there exists an isomorphism between the algebraic structure of Liouville theory coupled to $c = 1$ matter and the black hole CFT. To every state in $c = 1$ we can associate a state in the black hole CFT that has the same values of (p_ϕ, p_X). This isomorphism can be shown by realizing that there exists a correspondence between the complete energy-momentum tensors of both theories up to BRST-trivial terms. It gives a systematic way to obtain the representation of the physical states of the 2D black hole in terms of Wakimoto coordinates.

For $c = 1$ matter coupled to Liouville, the energy-momentum tensor (including the reparametrization ghosts) is

$$T_{c=1} = -\frac{1}{2}(\partial\phi)^2 - \sqrt{2}\partial^2\phi - \frac{1}{2}(\partial X)^2 - 2b\partial c - \partial bc. \tag{6.19}$$

In analogy, the total energy-momentum tensor of the Euclidean coset CFT can be written as

$$T_{\mathrm{SL}(2,\mathrm{IR})/\mathrm{U}(1)} = -\frac{1}{2}(\partial\phi')^2 - \sqrt{2}\partial^2\phi' - \frac{1}{2}(\partial X')^2 - 2b'\partial c - \partial b' c + \left\{Q^{\mathrm{total}}, -(\partial\log\gamma)B\right\}. \tag{6.20}$$

Here we have used the total BRST charge $Q^{\mathrm{total}} = Q^{\mathrm{U}(1)} + Q^{Diff}$ and we have defined the following transformation between the variables

$$\begin{aligned} \phi' &= \phi + \frac{1}{2\sqrt{2}}\log\gamma \\ X' &= X + i\frac{3}{2\sqrt{2}}\log\gamma \\ b' &= b + B\partial\log\gamma. \end{aligned} \tag{6.21}$$

The previous relations can be used to compute the physical states of the black hole CFT out of the states of $c = 1$ [24]. Of course one has to ensure that these states are indeed BRST-invariant. Using the transformations (6.21) and the form of the tachyon operators of $c = 1$ (6.1), this would imply that tachyon vertex operators in the black hole CFT have the form

$$\gamma^{-\frac{1}{2\sqrt{2}}(-3p_X+p_L)}\exp(ip_X X)\exp(p_L\phi), \tag{6.22}$$

and this is precisely the form of the dressed Kac-Moody primaries (4.15). A systematic analysis of the discrete states in the free-field approach can be done [24] if the transformation of variables (6.21) is used to obtain all the discrete states of the black hole in terms of the discrete states of $c = 1$. In the $c = 1$ model coupled to Liouville theory there appear operators at certain discrete values of the momenta [63] that form a W_∞ algebra. These operators have the form

$$W^+_{j,j} = \exp\left(i\sqrt{2}jX\right)\exp\left(\sqrt{2}(j-1)\phi\right) \qquad \text{with} \qquad j = 0, \frac{1}{2}, 1 \ldots, \tag{6.23}$$

$$W^+_{j,m} = \left(\oint \exp\left(-i\sqrt{2}X(w)\right)\right)^{j-m} W^+_{j,j} \qquad \text{with} \qquad -j \leq m \leq j. \tag{6.24}$$

There also exist discrete states on the wrong branch

$$W^-_{j,j} = \exp\left(i\sqrt{2}jX\right)\exp\left(-\sqrt{2}\,(j+1)\,\phi\right) \qquad \text{with} \qquad j = 0, \frac{1}{2}, 1 \ldots, \tag{6.25}$$

$$W^+_{j,m} = \left(\oint \exp\left(-i\sqrt{2}X(w)\right)\right)^{j-m} W^+_{j,j} \qquad \text{with} \qquad -j \leq m \leq j. \tag{6.26}$$

The corresponding operators in the black hole are obtained using the previous substitutions. Some examples are collected in ref.[24].

In this way it is possible to show the appearance of the W_∞ algebra and the ground ring elements that have been discovered in standard non-critical string theory [63,65] in the context of the black hole. The states obtained are all BRST-invariant [24]. All the physical states computed in ref. [24] can be formulated entirely in the $c = 1$ language i.e. in terms of the coordinates (ϕ, X). The β-γ contribution can be dropped out taking into account the BRST constraints and the bosonization formula for γ.

7. CORRELATION FUNCTIONS IN THE BLACK HOLE BACKGROUND

In this chapter we are going to compute the scattering amplitudes of tachyons in the black hole background and we will analyse the connection to the correlation functions of tachyon operators in standard non-critical string theory.

7.1 Scattering Amplitudes in String Theory

We are interested in the computation of a scattering amplitude

$$\mathcal{A}^{j_1 \ldots j_N}_{m_1 \ldots m_N} = \langle \mathcal{V}_{j_1\, m_1} \ldots \mathcal{V}_{j_N\, m_N} \rangle, \tag{7.1}$$

where the average has to be performed with respect to the action (4.14). We are going to consider spinless fields, i.e. winding modes so that we will use a short-hand notation where the antiholomorphic dependence with $m = \bar{m}$ will be understood. The tachyon vertex operators are described by the conformal invariant expression

$$\mathcal{V}_{j\, m} = \int \gamma^{j-m} \bar{\gamma}^{j-m} e^{\frac{2}{\alpha_+} j \phi} e^{im\sqrt{\frac{2}{k}} X} d^2 z \tag{7.2}$$

that satisfy the on-shell condition (3.46). The N-point correlation function of these vertex operators can be written as the following path integral

$$\int \prod_{i=1}^{N} d^2 z_i \frac{1}{\mathrm{Vol}_{\mathrm{SL}(2,\mathbb{C})}} \langle e^{im\sqrt{\frac{2}{k}} X(z_1,\bar{z}_1)} \ldots \rangle_X \langle e^{\frac{2}{\alpha_+} j \phi(z_1,\bar{z}_1)} \ldots \rangle_\phi \langle \gamma^{j_1 - m_1}(z_1) \ldots \rangle \langle \bar{\gamma}^{j_1 - m_1}(\bar{z}_1) \ldots \rangle_{\beta\gamma}. \tag{7.3}$$

In order to render the amplitude finite we have to divide out the volume of the group $\mathrm{SL}(2,\mathbb{C})$

$$\mathrm{Vol}_{\mathrm{SL}(2,\mathbb{C})} = \int \frac{d^2 z_1 d^2 z_2 d^2 z_3}{|z_1 - z_2|^2 |z_1 - z_3|^2 |z_2 - z_3|^2}. \tag{7.4}$$

Unfortunately, the computation of the above path integral is not an easy task since the involved field ϕ is not free and perturbation theory in M does not make sense. However, there is a clever way to circumvent this problem. After integrating out the zero modes of the fields we will see that we are left with the correlation functions of a free theory, and the interaction plays the role of new insertions.

7.2 Conservation Laws and Zero-Mode Integrations

We will now carry out the integration of the zero modes of the fields explicitly [66,58]. We start with the zero mode of ϕ. We introduce the notation $\phi = \widetilde{\phi} + \phi_0$, where ϕ_0 denotes the zero mode of the field. Vertex operators depending only on $\widetilde{\phi}$ will be denoted by a tilde. Using the identity

$$\exp\left(-M \int \beta\bar{\beta} e^{-\frac{2}{\alpha_+}\phi}\right) = \int_0^\infty dA\, \delta\left(\int \beta\bar{\beta} e^{-\frac{2}{\alpha_+}\phi} - A\right) e^{-MA} \tag{7.5}$$

we can write the ϕ-path integral in the following way

$$\begin{aligned}\left\langle \prod_{i=1}^N \mathcal{V}_{j_i\, m_i} \right\rangle = \left\langle \prod_{i=1}^N \widetilde{\mathcal{V}}_{j_i\, m_i} \right\rangle \int_0^\infty dA\, \delta\left(\int \beta\bar{\beta} e^{-\frac{2}{\alpha_+}\phi} - A\right) e^{-MA} \\ \times \int d\phi_0 \exp\left(\frac{\phi_0}{\alpha_+}\left(2\sum j_i + \frac{1}{\pi}\int R^{(2)}\right)\right).\end{aligned} \tag{7.6}$$

We introduce the notation $\tilde{A}$ for the surface integral of the screening operator depending only on $\tilde{\phi}$ and, using with some standard formulas for the δ-function, we find the expression

$$\delta\left(\int \beta\bar{\beta}e^{-\frac{2}{\alpha_+}\phi} - A\right) = -\frac{\alpha_+}{2A}\delta\left(\phi_0 - \frac{\alpha_+}{2}\log\left(\frac{\widetilde{A}}{A}\right)\right).$$

To integrate the zero-mode we take into account the Gauss-Bonnet theorem

$$\frac{1}{2\pi}\int R^{(2)}d^2z = (1-g), \tag{7.7}$$

where g is the genus of the Riemann surface. Inserting these conditions into the path integral we find the final result

$$\left\langle \prod_{i=1}^{N} \mathcal{V}_{j_i\, m_i} \right\rangle = M^s\Gamma(-s)\left\langle \prod_{i=1}^{N} \widetilde{\mathcal{V}}_{j_i\, m_i}\left(\int \beta\bar{\beta}e^{-\frac{2}{\alpha_+}\widetilde{\phi}}d^2z\right)^s \right\rangle_{M=0}, \tag{7.8}$$

where

$$s = \sum_{i=1}^{N} j_i + (1-g). \tag{7.9}$$

We have absorbed a factor $-\alpha_+/2$ into the definition of the path integral and we have used the identity

$$M^s\Gamma(-s) = \int_0^\infty dAA^{-s-1}e^{-MA}. \tag{7.10}$$

From the last expression we see that the amplitude is divergent if s is a positive integer. This divergence comes from the integration over the volume of the ϕ coordinate and can be regularized using the formula [34]

$$\mu^s \Gamma(-s)|_{s\to 0} \to \int_{\epsilon}^{\infty} \frac{dA}{A} e^{-\mu A} \to \log(\mu\epsilon). \tag{7.11}$$

This divergence has a nice space-time interpretation: the incoming particles are in resonance with s particles that form the wall against which they scatter.

From the previous considerations it becomes clear that the free-field approach is suitable to compute the amplitudes that obey a special energy sum rule where the number of screenings s is a positive integer. Any of the desired correlators can be determined indirectly by an analytic continuation in s as has been done with a tachyon background in standard Liouville theory [58,34,59]. From the zero mode of X we obtain with similar methods the conservation law

$$\sum_{i=1}^{N} m_i = 0. \tag{7.12}$$

The number of zero modes of the β-γ system, determined by the Riemann-Roch theorem, leads for spherical topologies to the condition:

$$\#\beta - \#\gamma = 1. \tag{7.13}$$

This constraint is equivalent to (7.9) and (7.12) for states of the form (4.15).

7.3 Scattering in the Bulk

We would like to study first the correlation functions that satisfy the charge conservation and therefore do not need screening charges, i.e. that satisfy $s = 0$. These correlators have been analysed in ref. [27,34]. It is clear that since we have no interaction for β and γ, this system does not contribute to the amplitudes; contractions coming from the γ's of the vertex operators (7.2) are equal to one. We will now see how the intermediate on-shell states produce poles in the amplitudes that are located at the positions where the discrete states of $c = 1$ occur. We introduce the notation

$$p_i = \sqrt{\frac{2}{k}}\, m_i \qquad \text{and} \qquad \beta_i = \frac{2}{\alpha_+} j_i, \tag{7.14}$$

in terms of which the tachyon operators (7.2) take the form

$$\mathcal{V}_p = \exp\left(ipX + \beta(p)\phi\right). \tag{7.15}$$

The γ part of the vertex operator can be dropped, since contractions of $\gamma's$ do not give any contribution to the scattering amplitudes if $s = 0$. In the above notation the on-shell condition (6.5) takes the form*

$$-\frac{\beta}{2}\left(\beta + Q\right) + \frac{p^2}{2} + \mathcal{N} = 1, \tag{7.16}$$

where $Q = 2/\alpha_+$ and $Q = 2\sqrt{2}$ for $k = 9/4$. The solution of the above equation has two branches

$$\beta = -\sqrt{2} \pm \sqrt{p^2 + 2\mathcal{N}}. \tag{7.17}$$

Operators with $\beta > -\sqrt{2}$ are called Seiberg states or operators on the right branch, while operators with $\beta < -\sqrt{2}$ are called anti-Seiberg states or operators on the wrong branch. When computing correlation functions of tachyon operators, we are going to consider operators on the right branch

---

$\star$ The parameter $\mathcal{N}$ appearing in this equation can be interpreted as the mass level, because the energy has the form $E = \beta + Q/2$ so that this equation is equivalent to the equation $E^2 = p^2 + m^2$.

$$\beta + \sqrt{2} = |p|, \tag{7.18}$$

in order to compare our results with the correlation functions of standard non-critical string theory. However, as we saw in section 5.2. the anti-Seiberg states play an important role in the physics of the 2D-black hole, since the black hole mass operator is a discrete state of this type. Therefore, in general, a correlation functions will include also operators of anti-Seiberg type that come from the screening charge.

For N tachyon operators the $s = 0$ amplitude is given by the Shapiro-Virasoro integral

$$\mathcal{A}_{s=0}(p_1, p_2, \ldots, p_N) = \int \prod_{i=4}^{N} d^2 z_i |z_i|^{2(p_1 p_i - \beta_1 \beta_i)} |1 - z_i|^{2(p_3 p_i - \beta_3 \beta_i)} \prod_{4 \le i < j \le N} |z_i - z_j|^{2(p_i p_j - \beta_i \beta_j)}. \tag{7.19}$$

In general, no closed expression for eq. (7.19) is known. The basic problem is the complicated pole structure of this integral; there are many channels in which poles appear. To analyse them we have to consider the region of the moduli integrals in eq. (7.19) where some of the z_i approach each other [34,67]. So for example, to analyse the limit $z_4, z_5, \ldots, z_{n+2} \to 0$ it is convenient to define the variables

$$z_4 = \varepsilon \qquad\qquad z_5 = \varepsilon y_5, \ldots, z_{n+2} = \varepsilon y_{n+2}, \tag{7.20}$$

and to consider the contribution of the region $|\varepsilon| << 1$ to the integral (7.19). There appear an infinite number of poles and the residues of these poles are related to correlation functions of on shell intermediate string states. In two space-time dimensions, it turns out that most of the residues of the above poles vanish, so that in this case one is able to obtain a closed expression for the amplitude. This is different from the situation in higher dimensions. This is why it is not possible to compute the above amplitude for $D \neq 2$. Plugging eq. (7.20) into eq. (7.19) we find explicit expressions for these poles. For $D = 2$ the first pole, for example, takes the form

$$\mathcal{A}(p_1, \ldots p_N) \simeq \frac{\langle \mathcal{V}_{p_1} \mathcal{V}_{p_4} \ldots \mathcal{V}_{p_{n+2}} \mathcal{V}_{-\widetilde{p}} \rangle \langle \mathcal{V}_{\widetilde{p}} \mathcal{V}_{p_2} \mathcal{V}_{p_3} \mathcal{V}_{p_{n+3}} \ldots \mathcal{V}_{p_N} \rangle}{\left(\sqrt{2} + \sum_i \beta_i\right)^2 - \left(\sum_i p_i\right)^2}, \tag{7.21}$$

where $\widetilde{p} = \sum_i p_i$ (where $i = 1, 4, 5, 6, \ldots, n+2$, because of momentum conservation). The pole in this amplitude indicates the appearance of an on shell intermediate tachyon with $\beta = \sum_i \beta_i$ and $p = \sum_i p_i$. It is simple to obtain an analog expression for the higher poles that are related to on shell intermediate states at higher mass levels.

Fig. 7: The factorization of the N-point tachyon amplitude by the OPE of the operators $4, \ldots, n+2$. The intermediate resonance particle has momentum $\widetilde{p}$.

A closed expression for these amplitudes has been computed in ref. [34,58,59,67]. Since the amplitudes are non-analytic in the momenta (7.18) the result is different in the different chirality configurations. If we consider a kinematical configuration where all tachyons except one have the same chirality, e.g. $(-, +, +, \ldots, +)$, then the momentum conservation for this kinematical configuration reads

$$p_1 + p_2 + \ldots + p_N = 0 \tag{7.22}$$

and the energy conservation is

$$-p_1 + p_2 + \ldots + p_N = \sqrt{2}(N-2) \tag{7.23}$$

This fixes (p_1, β_1) as a function of N

$$p_1 = \frac{2-N}{\sqrt{2}} \qquad\qquad \beta_1 = \frac{N-4}{\sqrt{2}} \tag{7.24}$$

We are now going to see that while p_1 is fixed by the kinematical constraints (7.24), the amplitude exhibits singularities as a function of the other momenta $p_2, p_3, \ldots, p_N$

$$p_i = \frac{\nu+1}{\sqrt{2}} \qquad \text{with} \qquad i = 2, \ldots, N \qquad \text{and} \qquad \nu = 0, 1, \ldots \tag{7.25}$$

but it has no singularities in combinations of the momenta.

To be more explicit, we consider the $s=0$ four-point amplitude of tachyons

$$\mathcal{A}_{s=0}(p_1, p_2, p_3, p_4) = \int d^2\eta |\eta|^{2\mathbf{p}_1\mathbf{p}_4} |1-\eta|^{2\mathbf{p}_3\mathbf{p}_4}, \tag{7.26}$$

where we have used the notations $\eta = z_{13}z_{24}/z_{12}z_{34}$, $\mathbf{p} = (p, -i\beta)$ and we have performed conformal transformations, to fix the positions of three vertices. The volume of SL$(2,\mathbb{C})$ has been dropped. The result of this integral can be computed exactly. For on-shell states it can be written in the form

$$\mathcal{A}_{s=0}(p_1,p_2,p_3,p_4) = \pi\Delta(1+\mathbf{p_1p_4})\Delta(1+\mathbf{p_2p_4})\Delta(1+\mathbf{p_3p_4}), \tag{7.27}$$

where $\Delta(x) = \Gamma(x)/\Gamma(1-x)$. We see that this amplitude has an infinite set of poles where

$$\mathbf{p_ip_4} = -\frac{1}{2}\left(\beta_i+\beta_4+\frac{Q}{2}\right)^2 + \frac{1}{2}(p_i+p_4)^2 - 1 = -\mathcal{N}-1, \tag{7.28}$$

for $i = 1$, 2 or 3. These are precisely the on-shell conditions for discrete states at level $\mathcal{N}$. If we choose the chirality configuration $(-,+,+,+)$ we obtain

$$\begin{aligned} j_i &= -\frac{1}{2}+\frac{2}{3}m_i \qquad \text{for} \qquad i=2,3,4 \\ j_1 &= -\frac{1}{2}-\frac{2}{3}m_1 \end{aligned} \tag{7.29}$$

After using the conservation laws and the condition $s = 0$, we have

$$(j_1, m_1) = (0, -3/2). \tag{7.30}$$

The amplitude takes the form

$$\widetilde{\mathcal{A}}^{j_1\ j_2\ j_3\ j_4}_{m_1\ m_2\ m_3\ m_4} = \pi\Delta\left(-4j_2-1\right)\Delta\left(-4j_3-1\right)\Delta\left(-4j_4-1\right). \tag{7.31}$$

As can be seen from the above formula the poles appear for $j = (\nu-1)/4$. These are precisely the $c = 1$ discrete states, in the classification (6.4).

We can obtain these poles as singularities from the OPE of two vertex operators that collide at one point. The short-distance singularities corresponding to $\eta \to 0$, i.e. $z_4 \to z_1$, appear if we expand the integrand of eq. (7.26) around $\eta = 0$. The result is

$$\mathcal{A}_{s=0}(p_1, p_2, p_3, p_4) \approx \int\limits_{|\eta|\leq\varepsilon} d^2\eta |\eta|^{2\mathbf{p_1 p_4}} \left| \sum_{\nu=0}^{\infty} \frac{\Gamma(1+\mathbf{p_3 p_4})}{\nu!\Gamma(1+\mathbf{p_3 p_4}-\nu)} (-\eta)^\nu \right|^2 . \tag{7.32}$$

This integral can be evaluated by transforming to polar coordinates. The result is

$$\mathcal{A}_{s=0}(p_1, p_2, p_3, p_4) \approx \sum_{\nu=0}^{\infty} \frac{\pi\varepsilon^{2(\nu+1+\mathbf{p_1 p_4})}}{\nu+1+\mathbf{p_1 p_4}} \left(\frac{\Gamma(1+\mathbf{p_3 p_4})}{\nu!\Gamma(1+\mathbf{p_3 p_4}-\nu)} \right)^2 . \tag{7.33}$$

If $\mathcal{V}_{p_3}$ is a generic tachyon, we see that the above amplitude has poles as a function of $\mathbf{p_1 p_4}$. These poles come from the contribution of operators in the OPE of $\mathcal{V}_{p_1}$ and $\mathcal{V}_{p_4}$, that satisfy eq. (7.28) for $i = 1$. The values of the external momenta $\mathbf{p_1}$ and $\mathbf{p_2}$ can be obtained from the kinematic relations. Choosing the chiralities of the four states as $(-, +, +, +)$ we obtain the relations

$$\begin{aligned} \mathbf{p_1} &= \left(-\sqrt{2}, 0\right) \qquad\qquad \mathbf{p_4} = \left(\frac{\nu+1}{\sqrt{2}}, -i\frac{\nu-1}{\sqrt{2}}\right) \\ \mathbf{p_j} &= \left(p_j, -i(\sqrt{2}+p_j)\right) \qquad \text{for} \qquad i = 2, 3. \end{aligned} \tag{7.34}$$

We observe that the pole structure of the exact result (7.28) is correctly reproduced by the poles that appear in the short distance singularities of two vertex operators [67]. The level of the intermediate state is equal to ν.

The above considerations can be generalized to N-point tachyon correlators without screening charges. The amplitudes again factorize in leg factors that have poles at all the discrete states belonging to the BRST cohomology of $c = 1$. Taking into account some symmetry properties of the correlator and the high energy behavior, the result for the N-point amplitude for the chirality configuration $(-, +, \ldots, +)$ is

$$\mathcal{A}(p_1, \ldots p_N) = \Gamma(N-1) \prod_{i=1}^{N} \Delta\left(\frac{\beta_i^2}{2} - \frac{p_i^2}{2}\right) \tag{7.35}$$

The amplitudes that contain two or more states in each chirality class vanish [34,67]. We conclude [27]: The scattering amplitudes in the black hole background that satisfy $s = 0$ have the same form as tachyon scattering amplitudes that satisfy $s = 0$ in standard non-critical string theory (7.35). In the next sections we are going to analyse the amplitudes with s different from zero [30,32] to see whether we find a correspondence with the amplitudes of standard non-critical string theory. We will start by considering the three-point function.

7.4 Three-Point Function with One Highest-Weight State

It turns out that the simplest way to address a general three-point tachyon amplitude is to take one state belonging to the discrete representation of SL(2,IR). We choose in this section one of the tachyons as a highest-weight state, for example $j_1 = m_1$. We will see later that an arbitrary three-point function, where no restriction is made to the representation to which the tachyons belong, can be expressed as a function of this one. The reason for this is that the SL(2,IR) Clebsch-Gordan coefficients can be analytically continued from one representation of SL(2,IR) to the others [50]. For the time being, the level of the Kac-Moody algebra is still arbitrary and we will need this restriction only when we want to make the analytic continuation in the number of screenings at the end.

Using SL(2,$\mathbb{C}$) transformations on the integrand, we can fix the three-tachyon vertex operators at $(z_1, z_2, z_3) = (0, 1, \infty)^\star$:

$$\widetilde{\mathcal{A}}^{j_1\ j_2\ \ j_3}_{j_1\ m_2\ m_3} = \int \prod_{i=1}^{s} d^2 z_i \big\langle e^{\frac{2}{\alpha_+} j_1 \widetilde{\phi}(0)} e^{\frac{2}{\alpha_+} j_2 \widetilde{\phi}(1)} e^{\frac{2}{\alpha_+} j_3 \widetilde{\phi}(\infty)} e^{-\frac{2}{\alpha_+} \widetilde{\phi}(z_i, \bar{z}_i)} \big\rangle$$
$$\langle \gamma^{j_2 - m_2}(1) \gamma^{j_3 - m_3}(\infty) \beta(z_i) \rangle \langle \bar{\gamma}^{j_2 - m_2}(1) \bar{\gamma}^{j_3 - m_3}(\infty) \bar{\beta}(\bar{z}_i) \rangle. \tag{7.36}$$

This correlator has the following form after bosonizing the β-γ system:

$$\widetilde{\mathcal{A}}^{j_1\ j_2\ \ j_3}_{j_1\ m_2\ m_3} = \int \prod_{i=1}^{s} d^2 z_i |z_i|^{-4\rho j_1} |1 - z_i|^{-4\rho j_2} \prod_{i<j} |z_i - z_j|^{4\rho} \mathcal{P}^{-1} \frac{\partial^s \mathcal{P}}{\partial z_1 \ldots \partial z_s} \bar{\mathcal{P}}^{-1} \frac{\partial^s \bar{\mathcal{P}}}{\partial \bar{z}_1 \ldots \partial \bar{z}_s}, \tag{7.37}$$

where

$$\mathcal{P} = \prod_{i=1}^{s} (1 - z_i)^{m_2 - j_2} \prod_{i<j} (z_i - z_j). \tag{7.38}$$

The derivatives of the above expression come from the bosonization of β (4.2). We can show that the following identity holds:

$$\mathcal{P}^{-1} \frac{\partial^s \mathcal{P}}{\partial z_1 \ldots \partial z_s} = \frac{\Gamma(-j_2 + m_2 + s)}{\Gamma(-j_2 + m_2)} \prod_{i=1}^{s} (1 - z_i)^{-1}. \tag{7.39}$$

The proof uses the definition of the Vandermonde:

$\star$ We will drop the zero mode in the next formulas.

$$\prod_{i=1}^{s}(z_i - z_j) = \sum_{\sigma(p(1),\ldots,p(s))} sign(p) z^0_{p(1)} \ldots z^{s-1}_{p(s)}, \tag{7.40}$$

where the sum goes over all permutations of the indices. Inserting this expression for the Vandermonde in the definition of $\mathcal{P}$ leads, after a simple calculation, to eq. (7.39). We introduce the notation $\rho = -2/\alpha_+^2 = -1/(k-2)$. The complete expression for the amplitude is:

$$\mathcal{A}^{j_1\ j_2\ \ j_3}_{j_1\ m_2\ m_3} = (-)^s \Delta(j_2 - m_2 + 1)\Delta(j_3 - m_3 + 1)\mathcal{I}(j_1, j_2, j_3, k). \tag{7.41}$$

We have used the identity:

$$\left(\frac{\Gamma(-j_2 + m_2 + s)}{\Gamma(-j_2 + m_2)}\right)^2 = (-)^s \Delta(j_2 - m_2 + 1)\Delta(j_3 - m_3 + 1), \tag{7.42}$$

which holds for integer s. The remaining integral:

$$\mathcal{I}(j_1, j_2, j_3, k) = M^s \Gamma(-s) \int \prod_{i=1}^{s} d^2 z_i |z_i|^{-4\rho j_1} |1 - z_i|^{-4\rho j_2 - 2} \prod_{i<j} |z_i - z_j|^{4\rho} \tag{7.43}$$

can be solved using the Dotsenko-Fateev formula (B.9) in ref. [57]. A careful treatment of the regularisation for the case $k = 9/4$ is needed. We will discuss this solution and the analytic continuation to arbitrary s later. From the above simple result, we can already see that SL(2,$\mathbb{R}$) fusion rules appear (see for example the appendix of ref. [23]). If the second tachyon is also in the highest-weight module, i.e. $m_2 = j_2 - \mathbb{N}$, then the result vanishes unless the conjugate of j_3 is also in the discrete representation i.e. $M = J - \mathbb{N}$, where $(J, M) = (-1 - j_3, -m_3)$. We will now see how this generalizes for an arbitrary three-point correlator, where a proportionality to a Clebsch-Gordan coefficient appears.

7.5 Three-Point Function of Arbitrary Tachyons

After getting an expression for the three-point function containing one highest-weight state and two generic tachyons, we would like to see how we can obtain the amplitude of three generic tachyons. Acting with the lowering operator $J_0^- = \oint J^-(z)dz$ we compute the amplitude $\mathcal{A}^{j_1}_{j_1-k_1}{}^{j_2}_{m_2}{}^{j_3}_{m_3}$, where the holomorphic $m_1 = j_1 - k_1$ dependence has been changed by an integer k_1. We will make an analytic continuation in k_1 to non-integer values, while for the time being s will be taken as an integer. Our computation shows that the general three-point function of (not necessarily on-shell) tachyons has the form:

$$\mathcal{A}^{j_1}_{m_1}{}^{j_2}_{m_2}{}^{j_3}_{m_3} = \mathcal{C}^2 \mathcal{I}(j_1, j_2, j_3, k), \tag{7.44}$$

where $\mathcal{C}$ is essentially the SL(2,$\mathbb{R}$) Clebsch-Gordan coefficient, whose expression we now calculate. The square takes the antiholomorphic $\bar{m}$ dependence into account. We will use the Baker-Campbell-Hausdorff formula:

$$e^{\alpha J_0^-}\mathcal{V}_{j\,m}e^{-\alpha J_0^-} = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}[J_0^-, \mathcal{V}_{j\,m}]_k, \tag{7.45}$$

where we have defined

$$[J_0{}^-, \mathcal{V}_{j\,m}]_0 = \mathcal{V}_{j\,m}, \qquad [J_0{}^-, \mathcal{V}_{j\,m}]_k = [J_0{}^-, [J_0{}^-, \mathcal{V}_{j\,m}]_{k-1}]. \tag{7.46}$$

The lowering operator J_0^- acts on the holomorphic part of the vertex operators:

$$[J_0^-, \mathcal{V}_{j\,m}] = -(j+m)\mathcal{V}_{j\,m-1}, \qquad [J_0^-, \mathcal{V}_{j\,m}]_k = (-)^k \frac{\Gamma(j+m+1)}{\Gamma(j+m-k+1)}\mathcal{V}_{j\,m-k}. \tag{7.47}$$

We are going to use the fact that J_0^- commutes with the screening charge $\mathcal{Q}$, which is actually only true up to a total derivative. The surface terms that appear are discussed in section 7.6. It is shown that they can be neglected in a particular region of (j_i, m_i), and the other regions can be obtained by analytic continuation [68]. With the above formulas and the fact that J_0^- annihilates the vacuum, we get an identity for the general amplitude as a function of that with one highest-weight state, if we identify in powers of α the r.h.s. and l.h.s. of:

$$\sum_{k_2,k_3=0}^{\infty} \frac{\alpha^{k_2+k_3}}{k_2!k_3!} \frac{\Gamma(j_2+m_2+1)}{\Gamma(j_2+m_2+1-k_2)} \frac{\Gamma(j_3+m_3+1)}{\Gamma(j_3+m_3+1-k_3)} \mathcal{A}^{j_1\ j_2\ \ \ \ \ \ \ \ j_3}_{j_1\ m_2-k_2\ m_3-k_3}$$
$$= \sum_{k_1=0}^{\infty} \frac{(-\alpha)^{k_1}}{k_1!} \frac{\Gamma(2j_1+1)}{\Gamma(2j_1+1-k_1)} \mathcal{A}^{j_1\ \ \ \ \ \ j_2\ \ j_3}_{j_1-k_1\ m_2\ m_3}. \tag{7.48}$$

Taking into account the antiholomorphic $\bar{m}$ dependence and formula (7.41) we get:

$$\mathcal{A}^{j_1\ \ \ \ \ \ j_2\ \ j_3}_{j_1-k_1\ m_2\ m_3} = \left(\frac{\Gamma(-2j_1)}{\Gamma(k_1-2j_1)} \sum_{n=0}^{k_1} \frac{1}{n!} \frac{\Gamma(k_1+1)}{\Gamma(k_1+1-n)} \frac{\Gamma(j_2+m_2+1)}{\Gamma(j_2+m_2+1-n)} \right.$$
$$\left. \frac{\Gamma(j_3+m_3+1)}{\Gamma(j_3+m_3+1-k_1+n)} \frac{\Gamma(-j_2+m_2-n+s)}{\Gamma(-j_2+m_2-n)} \right)^2 \mathcal{I}(j_1,j_2,j_3,k). \tag{7.49}$$

Our aim is to give up the condition that k_1 is an integer. First we notice that the above sum can be extended to ∞ and written in terms of the generalized hypergeometric function ${}_3F_2$ (A.5), which has a definition in terms of the Pochhammer double-loop contour integral that possesses a unique analytic continuation to the whole complex plane of all its indices [50]. Comparing eq. (7.44) and eq. (7.49):

$$\mathcal{C} = \frac{\Gamma(-2j_1)}{\Gamma(-j_1-m_1)} \frac{\Gamma(j_3+m_3+1)}{\Gamma(-j_2+m_2)} \frac{\Gamma(j_1+j_3+m_2+1)}{\Gamma(-j_1+j_3-m_2+1)}$$

$$\lim_{x\to 1} {}_3F_2(j_2-m_2+1, m_1-j_1, -j_2-m_2; -j_1-j_3-m_2, -j_1+j_3-m_2+1|x). \quad (7.50)$$

The appearance of the generalized hypergeometric function in our result is natural, since this function is always present in the theory of SL(2,IR) Clebsch-Gordan coefficients [50]. It can be expanded in s, which will be useful to check the analytic continuation in k_1 in simple examples, as we do in section 7.6. In general ${}_3F_2$ has a complicated expansion as a sum of Γ functions. Fortunately, for on-shell tachyons, which is the case we are interested in, we have a simple result.

We will consider three tachyons on the right branch satisfying $j \geq -1/2$. To fulfil the m conservation law (7.12), we take $m_2 \leq 0$ without loss of generality. If we choose $k = 9/4$ the on-shell conditions are (3.47):

$$2j_2 + 1 = -\frac{2}{3}m_2, \qquad 2j_i + 1 = \frac{2}{3}m_i \quad \text{for} \quad i = 1, 3. \qquad (7.51)$$

This fixes j_2 as a function of the screening,

$$j_2 = \frac{s}{2} - \frac{1}{4}, \qquad m_2 = -\frac{3}{2}s - \frac{3}{4}. \qquad (7.52)$$

With these kinematic relations it is easy to check that the generalized hypergeometric function is well-poised and we can apply Dixon's theorem [69] to simplify ${}_3F_2$. With (A.6) it is straightforward to obtain:

$$\mathcal{C}^2 = \rho^{2s} \left(\frac{\Gamma(-2j_1)}{\Gamma(-2j_1+s)}\right)^2 \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \prod_{i=1}^{3} \Delta\Big(2j_i + \frac{5}{4}\Big). \qquad (7.53)$$

We now have to evaluate the remaining part of the amplitude (7.44). The integral $\mathcal{I}(j_i,k)$ can be solved using the Dotsenko-Fateev (B.9) formula [57]:

$$
\begin{aligned}
\mathcal{I}(j_1,j_2,j_3,k) &= M^s\Gamma(-s)\int\prod_{i=1}^{s} d^2z_i|z_i|^{-4\rho j_1}|1-z_i|^{-4\rho j_2-2}\prod_{i<j}|z_i-z_j|^{4\rho} \\
&= \rho^{2s}(-\pi\Delta(1-\rho)M)^s\left(\frac{\Gamma(-2j_1+s)}{\Gamma(-2j_1)}\right)^2\mathcal{Y}_{13}\mathcal{Y}_2 \quad ,
\end{aligned}
\tag{7.54}
$$

where

$$
\mathcal{Y}_{13}=\prod_{i=0}^{s-1}\Delta(-2j_1\rho+i\rho)\Delta(-2j_3\rho+i\rho), \qquad \mathcal{Y}_2=\Gamma(-s)\Gamma(s+1)\prod_{i=0}^{s-1}\Delta((i+1)\rho)\Delta(-2j_2\rho+i\rho).
\tag{7.55}
$$

This result holds for arbitrary level k but integer screenings, so that it has to be transformed in order to obtain an expression valid for a non-integer s. This analytic continuation will be done à la Di Francesco and Kutasov [34], using the on-shell condition and taking the kinematics into account. From the above definition we notice that $\mathcal{Y}_2$ has dangerous singularities for $k=9/4$, i.e. $\rho=-4$, while $\mathcal{Y}_{13}$ is well defined since j_1 and j_3 could be chosen arbitrarily.

We first consider $\mathcal{Y}_{13}$. Choosing $j_1,j_3\notin\mathbb{Z}/8$ and the kinematic relations for $\rho=-4$, we obtain:

$$
\mathcal{Y}_{13}=\rho^{2s-2}\prod_{i=1,3}\Delta(-8j_i-2)\Delta\Big(2j_i+\frac{3}{4}\Big).
\tag{7.56}
$$

The product $\mathcal{Y}_2$ is more subtle because we have to find a regularisation that preserves the symmetries of the theory (see [33] for considerations related to this subject). In standard non-critical string theory this is achieved with the introduction of a background charge for the field X [34]. However, correlation functions involving discrete states are more delicate and the issues concerned with the regularisation are much more tricky (see for example [70,71]).

We will shift the level k away from the critical value and introduce a small parameter $\varepsilon \to 0$, which can be set to zero for $\mathcal{Y}_{13}$. This can be done by setting $\rho = -4 + 16\varepsilon$, i.e. $k = 9/4 + \varepsilon$, and taking the limit $\varepsilon \to 0$. With this modification of the level, the total central charge will be of order ε and the Liouville field $e^{\gamma\varphi_L}$ has to be taken into account in order to get an anomaly-free theory. For the on-shell condition we make the general ansatz:

$$2j + 1 = \pm\frac{2}{3}m + \varepsilon r(j, \gamma), \tag{7.57}$$

which in the limit $\varepsilon \to 0$ reduces our previous result. Here $r(j, \gamma)$ could be, in principle, an arbitrary function of the j's and the Liouville dressing γ. With the kinematics (7.57), we get:

$$s = 2j_2 + \frac{1}{2} + \frac{\varepsilon}{2}\widetilde{r} \tag{7.58}$$

with $\widetilde{r} = r(j_1, \gamma_1) - r(j_2, \gamma_2) + r(j_3, \gamma_3)$. After simple transformations we obtain:

$$\mathcal{Y}_2 = (-)^{s+1}\rho^{2s+1}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\Delta(-8j_2 - 2)\Delta\Big(2j_2 + \frac{3}{4}\Big). \tag{7.59}$$

In the above formula there appears a multiplicative factor:

$$\mathcal{R} = \frac{\Gamma(s + \frac{\widetilde{r}+4}{8})}{\Gamma(\frac{\widetilde{r}+4}{8})\Gamma(s+1)}, \tag{7.60}$$

which comes from the regularisation. We have no further constraint on the parameter $\widetilde{r}$ that appears in the above expression. The best we can do is to fix it by physical arguments. Choosing $\widetilde{r}=4$ will imply that the three-point function factorizes in leg factors and the four-point function will have obvious symmetry properties. This imposes strong constraints on $\widetilde{r}$. If we set $\widetilde{r}=4$ the contribution of the renormalization factor is one. We can gain more information about this renormalization by considering other correlation functions, as we will later do.

Our result for the on-shell tachyon three-point amplitude is obtained from eqs. (7.44), (7.54), (7.56) and (7.59):

$$\mathcal{A}(j_1,j_2,j_3)=-\rho^{6s-1}(-\pi M\Delta(-\rho))^s\prod_{i=1}^{3}\Delta(-8j_i-2)\Delta\Big(2j_i+\frac{3}{4}\Big)\Delta\Big(2j_i+\frac{5}{4}\Big), \tag{7.61}$$

which can be transformed finally to *:

$$\mathcal{A}(j_1,j_2,j_3)=\widetilde{M}^s\prod_{i=1}^{3}\Delta(-4j_i-1). \tag{7.62}$$

where $\widetilde{M}=-\pi M\Delta(-\rho)\rho^{-2}$.

We can translate our notation into the one used in $c=1$ where the β-γ system is eliminated. The vertex operators can be written as:

$$\mathcal{V}_p^{\pm}=e^{(-\sqrt{2}\pm|p|)\phi+ipX}. \tag{7.63}$$

Here $\pm$ denotes the tachyon vertex operators on the right (wrong) branch, which represent the incoming (outgoing) wave at infinity [72]. The three-point function of on-shell tachyons in the black hole background then takes the form:

$\star$ We absorb a factor of two in the definition of the path integral.

$$\mathcal{A}(p_1,p_2,p_3) = \widetilde{M}^s \prod_{i=1}^{3} \Delta(1-\sqrt{2}|p_i|), \qquad s = 1 + \frac{1}{2}\sum_{i=1}^{3}\left(\frac{|p_i|}{\sqrt{2}} - 1\right). \quad (7.64)$$

This can be compared with the $c = 1$ three-point function, with tachyonic background [34]:

$$A(p_1,p_2,p_3) = [\mu\Delta(-\rho)]^s \prod_{i=1}^{3}(-\pi)\Delta(1-\sqrt{2}|p_i|), \qquad \frac{s}{2} = 1 + \frac{1}{2}\sum_{i=1}^{3}\left(\frac{|p_i|}{\sqrt{2}} - 1\right). \quad (7.65)$$

There appear several remarkable features in our result:

- ▹ In order to get finite correlation functions, the parameter M has been infinitely renormalized, as is done for the cosmological constant μ in Liouville theory. Here it is known to be equivalent to the replacement $e^{-\sqrt{2}\varphi} \rightarrow \varphi e^{-\sqrt{2}\varphi}$, which may have interesting physical consequences. Perhaps this will be the case for the black hole model as well.
- ▹ From the zero mode integration of the N-point function in both theories, we see that the number of screenings for the black hole is half of that in the $c = 1$ model.
- ▹ The amplitude can be factorized in leg poles, which (with this normalization) have resonance poles where the $c = 1$ discrete states are placed. The new discrete states of Distler and Nelson do not appear. The explanation of this is that these extra states are BRST-trivial in the Wakimoto representation. This has been shown by Bershadsky and Kutasov for the first examples, as we saw in section 5.2 [27].

7.6 Illustrative Example

As an illustrative example we will consider in more detail the three-point function with one screening. In this case we have:

$$m_1 + m_2 + m_3 = j_1 + j_2 + j_3 = 0 \tag{7.66}$$

and the correlator is given by:

$$\mathcal{A}^{j_1\ j_2\ j_3}_{m_1\ m_2\ m_3} = \int d^2 z\ \langle \mathcal{V}_{j_1\ m_1}(0)\mathcal{V}_{j_2\ m_2}(1)\mathcal{V}_{j_3\ m_3}(\infty)\ \beta(z)\bar{\beta}(\bar{z})\, e^{-\frac{2}{\alpha_+}\phi(z,\bar{z})}\rangle_{M=0}$$

$$= M\Gamma(-s)\int d^2 z\ \left(\frac{j_1 - m_1}{z} - \frac{j_2 - m_2}{1-z}\right)\left(\frac{j_1 - m_1}{\bar{z}} - \frac{j_2 - m_2}{1-\bar{z}}\right)|z|^{-4\rho j_1}\,|1-z|^{-4\rho j_2}. \tag{7.67}$$

To evaluate this integral directly, without any restriction on the m dependence of the three vertex operators, we have to use partial integration. We get:

$$\mathcal{A}^{j_1\ j_2\ j_3}_{m_1\ m_2\ m_3} = M\Gamma(-s)\left(\frac{-m_1 j_2 + m_2 j_1}{j_1}\right)^2 \int d^2 z |z|^{-4\rho j_1}|1-z|^{-4\rho j_2 - 2} + \mathcal{B}(j_1, j_2) + \mathcal{B}^*(j_1, j_2) \tag{7.68}$$

where $\mathcal{B}(j_1, j_2)$ is the part coming from the boundaries of the region of integration. It is proportional to:

$$\mathcal{B}(j_1, j_2) \sim \int d^2 z \frac{\partial}{\partial \bar{z}}\Big(|z|^{-4\rho j_1}|1-z|^{-4\rho j_2 - 2}(1-\bar{z})\Big). \tag{7.69}$$

This integral can be evaluated using (see appendix A of ref. [73])

$$\int_{\Sigma} d^2 z \frac{\partial}{\partial \bar{z}} (f(z,\bar{z})) = -\frac{i}{2} \int_{\partial\Sigma} dz f. \tag{7.70}$$

We obtain:

$$\mathcal{B}(j_1, j_2) \sim -\frac{i}{2} \lim_{\varepsilon \to 0} \oint_{\varepsilon} dz\, |z|^{-4\rho j_1} \, |1-z|^{-4\rho j_2 - 2} \, (1-\bar{z}) = \pi \lim_{\varepsilon \to 0} \varepsilon^{-8\,\rho j_2}, \tag{7.71}$$

where the integral is around a small circle of radius ε around 1. The contribution of the surface term is zero for $j_2 > 0$, finite for $j_2 = 0$, and diverges for $j_2 < 0$. As argued by Green and Seiberg [68], a finite contact term must be added in the case where the boundary terms are finite and an infinite term if they diverge in order to render the amplitude analytic. These contact terms can be avoided by calculating amplitudes in an appropriate kinematic configuration, where the contact terms are not needed, and then analytically continuing to the desired kinematics. For $s > 1$, our argumentation will be the same as for $s = 1$, and we will restrict the values of j to the regions where the boundary terms vanish. This also means that we will restrict to the kinematic regions where J_0^- commutes with the screening charge. We can compare this result with the one, following from eq. (7.44), which is based on the analytic continuation in $k_1 = j_1 - m_1$ to non-integer values. We obtain $\mathcal{C}$ for integer screenings by expanding eq. (7.50):

$$\begin{aligned} \mathcal{C} =&(-)^s \frac{\Gamma(-2j_1)\Gamma(s+1)}{\Gamma(-2j_1+s)} \sum_{i=0}^{s} \frac{\Gamma(m_1 - j_1 + i)}{\Gamma(m_1 - j_1)} \frac{\Gamma(-m_2 - j_2 + i)}{\Gamma(-m_2 - j_2)} \\ & \frac{\Gamma(-m_1 - j_1 + s - i)}{\Gamma(-m_1 - j_1)} \frac{\Gamma(m_2 - j_2 + s - i)}{\Gamma(m_2 - j_2)} \frac{(-)^i}{(s-i)!i!}, \end{aligned} \tag{7.72}$$

which for $s = 1$ gives:

$$\mathcal{C}_{s=1} = \frac{j_1 m_2 - m_1 j_2}{j_1}. \tag{7.73}$$

With eq. (7.44) and eq. (7.43) the amplitude becomes eq. (7.68), where $\mathcal{B}(j_1, j_2) = 0$. With this explicit example, one can already see that the analytic continuation in k_1 is correct. The integral can be solved using (A.7) for $m = 1$ and the result is eq. (7.62).

In the next section we will calculate the two-point function to see whether similar characteristics appear. If we have $c \leq 1$ matter coupled to Liouville theory the simplest way to construct a two-point function of on-shell tachyons is to use the three-point function. One of the operators is then set to be the dressed identity and this is the derivative with respect to the cosmological constant μ of the two-point function [34]. In this case the situation is different because the interaction is not a tachyon but a discrete state of $c = 1$ [24,27]. Fortunately we can construct the two-point function of (not necessarily on-shell) tachyons in the black hole background. We will do this with two independent methods, as a double check of our computation.

7.7 The Two-Point Function

We can perform a direct computation, fixing the position of one of the screenings at $z_s = \infty$ and evaluating the remaining $(s-1)$ integrals:

$$\langle \mathcal{V}_{j_1\, m_1}(0) \mathcal{V}_{j_2\, m_2}(1) \rangle = \lim_{z_s \to \infty} \langle \mathcal{V}_{j_1\, m_1}(0) \mathcal{V}_{j_2\, m_2}(1)\ \beta(z_s) \bar{\beta}(\bar{z}_s) e^{-\frac{2}{\alpha_+}\phi(z_s, \bar{z}_s)}\ \mathcal{Q}^{s-1} \rangle. \tag{7.74}$$

We can follow closely the steps of the previous computation, although in this case it will be much simpler. We use the representation (4.9) for the vertex operators and the bosonization formulas for the β-γ system. The zero mode integrations give:

$$s = j_1 + j_2 + 1, \qquad\qquad m_1 + m_2 = 0. \tag{7.75}$$

Due to the kinematic relations satisfied by the two-point amplitude, the part of the integrand coming from the β-γ system can be written as:

$$\mathcal{P}^{-1}\frac{\partial^s \mathcal{P}}{\partial z_1 \ldots \partial z_s}\bar{\mathcal{P}}^{-1}\frac{\partial^s \bar{\mathcal{P}}}{\partial \bar{z}_1 \ldots \partial \bar{z}_s} = (-)^s\, \Delta(1{+}j_1{-}m_1)\Delta(1{+}j_2{-}m_2) \prod_{i=1}^{s} |z_i|^{-2}\, |1{-}z_i|^{-2}, \tag{7.76}$$

where

$$\mathcal{P} = \prod_{i=1}^{s} z_i^{m_1-j_1}(1-z_i)^{m_2-j_2} \prod_{i<j}^{s}(z_i - z_j). \tag{7.77}$$

This can easily be seen with the substitution $z_i = 1/y_i$. After evaluating the ϕ contractions and taking $z_s \to \infty$, the complete $\mathcal{A}_{1\to 1}$ amplitude is reduced to the evaluation of the following integral:

$$\begin{aligned}\langle \mathcal{V}_{j_1\, m_1}(0)\mathcal{V}_{j_2\, m_2}(1)\rangle =& (-)^s M^s \Gamma(-s)\Delta(1+j_1-m_1)\Delta(1+j_2-m_2) \\ & \int \prod_{i=1}^{s-1} d^2 z_i |z_i|^{-4\rho j_1 - 2}|1-z_i|^{-4\rho j_2 - 2} \prod_{i<j}^{s-1} |z_i - z_j|^{4\rho}.\end{aligned} \tag{7.78}$$

The result is well defined for $\rho \neq -4$ and can be obtained from (A.6):

$$\langle \mathcal{V}_{j_1\, m_1}(0)\mathcal{V}_{j_2\, m_2}(1)\rangle = (-)^s M^s \Gamma(-s)\Delta(1+j_1-m_1)\Delta(1+j_2-m_2)\Gamma(s)$$

$$(\pi\Delta(1-\rho))^{s-1}\prod_{i=1}^{s-1}\Delta(i\rho)\prod_{i=0}^{s-2}\Delta(-2j_1\rho+i\rho)\Delta(-2j_2\rho+i\rho)\Delta(1+\rho(s-i)). \quad (7.79)$$

In the next section we show that the two-point function of (not necessary) on-shell tachyons is only different from zero for $j_1 = j_2 = j$, so that we set $s = 2j+1$ and, from the conservation law $m_1 = -m_2 = m \geq 0$, we obtain for an arbitrary level:

$$\langle \mathcal{V}_{j\, m}(0)\mathcal{V}_{j\, -m}(1)\rangle = (-\pi M\Delta(-\rho))^s \Delta(1+j-m)\Delta(1+j+m)s\Delta(1-s)\Delta(\rho s). \quad (7.80)$$

If $k \neq 9/4$ and the tachyons do not belong to a discrete representation of SL(2,$\mathbb{R}$), the above amplitude has one divergence, which appears for s integer and comes from the zero mode integration. For $k = 9/4$ we demand $j \notin \mathbb{Z}/2$, which implies that s is non-integer. This can always be done, since the above expression is well defined in this case. The final expression for on-shell tachyons is:

$$\langle \mathcal{V}_{j\, m}(0)\mathcal{V}_{j\, -m}(1)\rangle = \frac{\widetilde{M}^{2j+1}}{2j+1}(\Delta(-4j-1))^2, \quad (7.81)$$

where $\widetilde{M}$ is the renormalized black hole mass defined previously.

We can obtain the answer from the three-point tachyon amplitude that contains one highest-weight state $\mathcal{V}_{j_1\, j_1}$, taking the limit $j_1 = i\varepsilon \to 0$. In this way we are fixing only two points of the SL(2,$\mathbb{C}$)-invariant $\mathcal{A}_{1\to 1}$ amplitude. The result will contain a divergence coming from the volume of the dilation group [61,62].

First we can show that this amplitude is diagonal by pushing J_0^+ and J_0^- through the correlator [50]. Here again we will use the fact that the screening charge commutes with the currents. We obtain the relations

$$\frac{\langle \mathcal{V}_{j_2\ m_2}\mathcal{V}_{j_3\ m_3+1}\rangle}{\langle \mathcal{V}_{j_2\ m_2+1}\mathcal{V}_{j_3\ m_3}\rangle} = -\frac{j_2-m_2}{j_3-m_3} = -\frac{j_3+m_3+1}{j_2+m_2+1}. \tag{7.82}$$

From the X zero-mode integration we get $m_2 = m$ and $m_3 = -1-m$, so that this equation has two solutions, one with $j_2 = -1-j_3$, which contains no screenings, and another one with $j_2 = j_3$, which is a two-point function with $s = 2j_2+1$ screenings. The first one is normalized to one up to the divergence $\Gamma(0)$ coming from the zero mode integral. We now compute the two-point function with screenings.

We first consider the case of arbitrary k and take the limit $k \to 9/4$ at the end of the calculation. From eq. (7.41), we obtain for $j_1 = i\varepsilon$:

$$\mathcal{A}^{j_2\ \ j_3}_{m_2\ m_3} = (-)^s \Delta(1+j_2-m_2)\Delta(1+j_3-m_3) I(j_2,j_3,\rho), \tag{7.83}$$

where

$$\begin{aligned} I(j_2,j_3,k) &= M^s\Gamma(-s)\lim_{\varepsilon\to 0}\int\prod_{n=1}^{s} d^2z_n|z_n|^{-4\rho\varepsilon i}|1-z_n|^{-4\rho j_2-2}\prod_{n<m}|z_n-z_m|^{4\rho} \\ &= -(\pi M\Delta(-\rho))^s\Delta(1-s)\Delta(\rho s)\lim_{\varepsilon\to 0}\Delta(1-2\rho\varepsilon i)\Delta((\varepsilon i-j_2+j_3)\rho))\Delta((\varepsilon i+j_2-j_3)\rho). \end{aligned} \tag{7.84}$$

Here we have simplified the products of well defined Δ-functions. To evaluate the limit we take (A.4) and the following representation of the δ-distribution:

$$\delta(j_2 - j_3) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + (j_2 - j_3)^2}. \tag{7.85}$$

into account. In total we obtain for the two-point function of generic (in general off-shell) tachyons:

$$\mathcal{A}^{j_2\ j_3}_{m_2\ m_3} = 2\pi i \rho \delta(j_2 - j_3)(-\pi M \Delta(-\rho))^s \Delta(1-s)\Delta(\rho s)\Delta(1+j_2-m_2)\Delta(1+j_3-m_3), \tag{7.86}$$

where $s = 2j_2 + 1$ and the level is arbitrary. This agrees with eq. (7.80) up to a factor s and the volume of the dilation group $\delta(j_2 - j_3)$.

Using the $c = 1$ language, we obtain for the two-point function (7.81) of two Seiberg on-shell tachyons in the black hole background*

$$\langle \mathcal{V}^+_p(0) \mathcal{V}^+_{-p}(1) \rangle = \frac{\widetilde{M}^{\frac{|p|}{\sqrt{2}}}}{\sqrt{2}|p|} (\Delta(1 - \sqrt{2}|p|))^2, \tag{7.87}$$

The two-point function in standard non-critical string theory is:

$$\langle V^+_p(0) V^+_{-p}(1) \rangle = \frac{\mu^{\sqrt{2}|p|}}{\sqrt{2}|p|} (\Delta(1 - \sqrt{2}|p|))^2. \tag{7.88}$$

Comparing with the two-point function in the black hole background we find the same features as for the three-point function. The pole structure is the same as the one of the two-point function of tachyons of $c = 1$ at non-vanishing cosmological constant, while the screenings differ by a factor of 2.

The two-point function of the deformed matrix model is given by the expression [16]:

$\star$ We have absorbed a factor of 2 as before.

$$\langle V_p^+(0)V_{-p}^+(1)\rangle \sim \frac{M^{\frac{|p|}{\sqrt{2}}}}{\sqrt{2}|p|}\Delta\left(1-\frac{|p|}{\sqrt{2}}\right)^2. \tag{7.89}$$

Here only half of the states of $c=1$ (the supplementary series) appear as poles in the leg factors. Both two-point functions can be reconciled if we take into account that we can renormalize the tachyon vertex operators in a different way according to the SL(2,$\mathbb{R}$) representation theory. If in the normalization (3.19) we use the on shell condition and we take into account the antiholomorphic piece, we renormalize our operators as follows:

$$\mathcal{V}_p^+ \to \frac{\mathcal{V}_p^+}{\Delta\left(\frac{1}{2}-\frac{|p|}{\sqrt{2}}\right)} \tag{7.90}$$

Here a regular function depending on p has been dropped, which is not determined by eq. (3.19). The two-point function of these operators in the black hole background agrees with eq. (7.89). It is simple to see how the N-point function of these differently normalized tachyons behaves. The N-point function with chirality $(+,\ldots,+,-)$, will be divided by a factor $\Delta(-s-(N-1)/2)$, coming from the state with the opposite chirality. This will imply, that $(+,\ldots,+,-)$ odd-point functions of these operators vanish for positive integer screenings, if they were previously finite. For even point functions this factor is of course irrelevant for integer s so that they are finite in this case. Which one is the correct physical normalization is to be clarified. For related problems see ref. [62].

7.8 The Four-Point Function

In the previous sections we have seen that we are able to obtain closed expressions for the two- and three-point tachyon correlation functions in the Euclidean black hole background. Now we are going to consider the four-point amplitude of tachyons[†]. As in the computation of scattering amplitudes of minimal models coupled to Liouville theory [34], we are faced with the problem that the integrals that have to be computed do not exist in the mathematical literature. However, we are able to address this problem with similar methods as those used in refs. [34,67] (see ref. [74] for a review), although in this case the situation is more complicated. The three-point function of on-shell tachyons previously computed will be one of the basic ingredients. The results of this section should be considered as preliminary and will be discussed in more detail in ref. [30]. A general four-point function has the form

$$\widetilde{\mathcal{A}}^{j_1\ j_2\ j_3\ j_4}_{m_1\ m_2\ m_3\ m_4} = \int d^2\eta |\eta|^{2\mathbf{p}_1\mathbf{p}_4}|1-\eta|^{2\mathbf{p}_3\mathbf{p}_4} \int \prod_{i=1}^{s} d^2 z_i \left(\mathcal{P}^{-1} \frac{\partial^s \mathcal{P}}{\partial z_1 \ldots \partial z_s} \bar{\mathcal{P}}^{-1} \frac{\partial^s \bar{\mathcal{P}}}{\partial \bar{z}_1 \ldots \partial \bar{z}_s} \right)$$
$$\times \prod_{i=1}^{s} |z_i|^{-4\rho j_1} |1-z_i|^{-4\rho j_3} |\eta - z_i|^{-4\rho j_4} \prod_{i<j} |z_i - z_j|^{4\rho}, \tag{7.91}$$

where

$$\mathcal{P} = \prod_{i=1}^{s} z_i^{m_1 - j_1} (1-z_i)^{m_3 - j_3} (\eta - z_i)^{m_4 - j_4} \prod_{i<j} (z_i - z_j). \tag{7.92}$$

We have fixed the positions of the four vertex operators at $(z_1, z_2, z_3, z_4) = (0, \infty, 1, \eta)$. The above integral is very complicated and to get a closed expression for the solution we will use

- ▹ the symmetries,
- ▹ the pole structure and
- ▹ the high energy behavior

† We would like to thank U. Danielsson for discussions on this subject.

of the amplitude. These properties will be enough to determine the form of the solution using some powerful theorems for entire functions. We will compute the amplitude for $j_i > 0$ and can obtain more general amplitudes using analytic continuation in j_i. We will begin with the determination of the symmetries.

Symmetries

The integral (7.91) exhibits several symmetries. Changing the integration variables to $\eta \to 1 - \eta$ and $w_i \to 1 - w_i$ we obtain

$$\widetilde{\mathcal{A}}^{j_1\ j_2\ j_3\ j_4}_{m_1\ m_2\ m_3\ m_4} = \widetilde{\mathcal{A}}^{j_3\ j_2\ j_1\ j_4}_{m_3\ m_2\ m_1\ m_4}. \tag{7.93}$$

With a change of variables of the form $\eta \to 1/\eta$ and $w_i \to 1/w_i$ we obtain the symmetry

$$\widetilde{\mathcal{A}}^{j_1\ j_2\ j_3\ j_4}_{m_1\ m_2\ m_3\ m_4} = \widetilde{\mathcal{A}}^{-j_1-j_3-j_4-1+s\ \ j_2\ j_3\ j_4}_{-m_1-m_3-m_4\ \ \ \ m_2\ m_3\ m_4}. \tag{7.94}$$

Pole Structure

The amplitude (7.91) has a complicated pole structure. These poles can be computed by evaluating the short-distance singularities that appear if some vertex operators (and some screening charges) collide at one point. In these arguments the screening charge has to be treated carefully, because it is not a tachyon operator but a discrete state of $c = 1$. In the case where the perturbation is a cosmological constant we can compute the N-point function with s screenings out of the $(N+s)$-point function without screenings, by sending s of the momenta to zero. In the black hole we have to distinguish between tachyon operators as external insertions and the screening charge that is a discrete state. We will begin by analysing the poles that appear if the vertex operator $\mathcal{V}_{j_4\ m_4}(\eta)$ approaches $\mathcal{V}_{j_1\ m_1}(0)$, while leaving the screenings far away from zero. The other possibilities, where η approaches 1 or ∞, are equivalent. The OPE of two vertex operators takes the form:

$$: e^{i\mathbf{p_4}\cdot\mathbf{X}(\eta)} :: e^{i\mathbf{p_1}\cdot\mathbf{X}(0)} := \sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)^2 |\eta|^{2\mathbf{p_1}\cdot\mathbf{p_4}+2n} V_n(0) \tag{7.95}$$

where $\mathbf{X}=(X,\phi)$ and we have taken the same for the holomorphic and the anti-holomorphic part since these are the only contributions when we integrate over η. The operators on the r.h.s. are given by

$$V_n =: e^{i\mathbf{p_4}\cdot\mathbf{X}}\partial^n\bar{\partial}^n e^{i\mathbf{p_1}\cdot\mathbf{X}} :=: \left(-\mathbf{p_1}\cdot\partial^n\mathbf{X}\mathbf{p_1}\cdot\bar{\partial}^n\mathbf{X}+\ldots\right)e^{i\mathbf{p}\cdot\mathbf{X}} :, \tag{7.96}$$

where $\mathbf{p}=\mathbf{p_1}+\mathbf{p_4}$. In the black hole CFT, the OPE between two vertex operators of the form (4.15) is

$$\mathcal{V}_{j_4\ m_4}(\eta,\bar{\eta})\mathcal{V}_{j_1\ m_1}(0,0) = |\eta|^{2\mathbf{p_1}\cdot\mathbf{p_4}}\mathcal{V}_{j_1+j_4\ m_1+m_4}(0,0)+\sum_{n=1}^{\infty}|\eta|^{2\mathbf{p_1}\cdot\mathbf{p_4}+2n}\mathcal{V}_n(0,0). \tag{7.97}$$

All the operators from the r.h.s. satisfy $(J,M)=(j_1+j_4,m_1+m_4)$. The first operator is a discrete tachyon that appears at zero mass level, while $\mathcal{V}_n$ denote the discrete states that appear at higher mass levels.

To analyze the pole structure we insert the OPE (7.97) into the four-point function and near the poles that come from $\eta\to 0$ we obtain

$$\widetilde{\mathcal{A}}^{j_1\ j_2\ j_3\ j_4}_{m_1\ m_2\ m_3\ m_4} \approx \int\limits_{|\eta|\leq\varepsilon} d^2\eta|\eta|^{2\mathbf{p}_1\mathbf{p}_4}\langle\mathcal{V}_{j_1+j_4\ m_1+m_4}(0)\mathcal{V}_{j_2\ m_2}(\infty)\mathcal{V}_{j_3\ m_3}(1)\rangle+$$
$$\sum_{n=1}^{\infty}\int\limits_{|\eta|\leq\varepsilon} d^2\eta|\eta|^{2\mathbf{p}_1\mathbf{p}_4+2n}\langle\mathcal{V}_n(0)\mathcal{V}_{j_2\ m_2}(\infty)\mathcal{V}_{j_3\ m_3}(1)\rangle. \tag{7.98}$$

The residues of the poles of the above integral are described by three-point functions that, in general, contain one discrete state and two tachyon operators as external states. The poles appear if the integral over η diverges for $\varepsilon \approx 0$. To evaluate the integral we use polar coordinates and obtain

$$\int\limits_{|\eta|\leq\varepsilon} d^2\eta|\eta|^{2\mathbf{p}_1\mathbf{p}_4+2n} \approx \pi\frac{\varepsilon^{2\mathbf{p}_1\mathbf{p}_4+2n+2}}{\mathbf{p}_1\mathbf{p}_4+n+1}. \tag{7.99}$$

Therefore, the poles appear at

$$\frac{2}{k}m_1m_4 - \frac{4}{\alpha_+^2}j_1j_4 + n + 1 = 0. \tag{7.100}$$

The above equation is precisely the condition that the intermediate state is on-shell:

$$-\frac{(j_1+j_4)(j_1+j_4+1)}{k-2} + \frac{(m_1+m_4)^2}{k} + n = 1, \tag{7.101}$$

where the mass-level of the intermediate particle is $\mathcal{N} = n$. We consider first the chirality configuration $(+,+,+,-)$. We are interested in the case $k = 9/4$; the kinematical relations are then

$$2j_i + 1 = \frac{2}{3} m_i \qquad \text{for} \qquad i = 1, 2, 3$$
$$2j_4 + 1 = -\frac{2}{3} m_4. \tag{7.102}$$

Using the above kinematics we obtain the relations

$$j_4 = \frac{s}{2} \qquad \text{and} \qquad j_2 = -j_1 - j_3 - 1 + \frac{s}{2}. \tag{7.103}$$

This means that the independent kinematic variables that describe this process are j_1, j_3 and s. The poles appear in the amplitude if

$$j_1 = \frac{\mathcal{N}}{4(2s+1)} - \frac{1}{4}. \tag{7.104}$$

Since the intermediate state satisfies $(J, M) = (j_1 + j_4, m_1 + m_4)$ we obtain from eqs. (7.103) and (7.104):

$$J = j_1 + j_4 = \frac{1}{4} \left(\frac{\mathcal{N}}{2s+1} - 1 + 2s \right). \tag{7.105}$$

We will first analyse the case where the intermediate state is a discrete tachyon:

$$J = j_1 + j_4 = \frac{2s-1}{4}. \tag{7.106}$$

The corresponding four-point function is represented in Fig. 8. While the first three-point function contains s screenings, the second three-point function satisfies $s = 0$ and is therefore equal to one.

Fig. 8: Near the pole, the four-point amplitude with a chirality configuration $(+,+,+,-)$ can be factorized into two three-point functions. The intermediate state is on-shell, has negative chirality and is on the right branch.

Using the kinematical relations (7.102) and (7.106) we obtain that one of the external states is fixed at $(j_1, m_1) = (-1/4, 3/4)$, while $(j_4, m_4) = (s/2, -3(s+1)/2)$. Since the intermediate state has negative chirality the residuum of the pole is described by the three-point function:

$$\langle \mathcal{V}^{-}_{j_1+j_4\ m_1+m_4}(0)\mathcal{V}^{+}_{j_2\ m_2}(\infty)\mathcal{V}^{+}_{j_3\ m_3}(1)\rangle, \tag{7.107}$$

where the upper index denotes the chirality. Using the expression for the on-shell three-point function (7.62) it is easy to see that this residuum is different from zero. We conclude that for the value of J, given in equation (7.106), there appears a pole in the four-point function that corresponds to an intermediate discrete tachyon.

We now consider the case where the intermediate particle is a state at higher mass level and we would like to determine those values of J from eq. (7.105) for which there appear poles in the amplitude. First we observe from eq. (7.105) that the new discrete states of Distler and Nelson [22] do not appear. To obtain the poles, we can use the arguments of ref. [67] for $c = 1$ based on the decoupling of null-states. Only if $\mathcal{N} = \alpha(2s+1)$, where α is a positive integer does the intermediate state describe an on-shell physical string state of the classification (6.4)

$$J = \frac{\alpha - 1}{4} + \frac{s}{2} \qquad \text{with} \qquad \alpha \in \mathbb{N}. \tag{7.108}$$

In this case we have $j_1 = (\alpha - 1)/4$.

To make clear the above condition we consider a simple example [67]. Using the $c = 1$ language [24,27] the on-shell vertex operator of the black hole at level $\mathcal{N} = 1$ can be written in the form

$$\mathcal{V}_1 = \mathbf{p} \cdot \partial \mathbf{X} \mathbf{p} \cdot \bar{\partial} \mathbf{X} e^{i\mathbf{p}\cdot\mathbf{X}} = -\partial\bar{\partial} e^{i\mathbf{p}\cdot\mathbf{X}}. \tag{7.109}$$

This state is clearly a null state for generic values of $\mathbf{p}$, so that it decouples from correlation functions and the residuum vanishes. The situation is different for $\mathbf{p} = 0$. Then this state has $(J, M) = (0, 0)$ and corresponds to a discrete state. It appears as a pole in amplitudes that satisfy $s = 0$, where $\alpha = 1$. Similarly we can argue for the other residues. They are described by three-point functions involving one state at higher mass-level and vanish if this state is not an on-shell physical string state described by eq. (7.108). This decoupling of null states is a property of tachyon correlation functions of $c = 1$ matter coupled to Liouville theory and it is plausible that the same property is satisfied by the black hole CFT as well.

We can carry out a similar analysis of the pole structure by scaling a number r of screening charges to zero as $\eta \to 0$. It turns out that the poles occur for $2j_1$ integer or half-integer

$$j_1 = \frac{n}{4(2(s-r)+1)} + \frac{r}{2} - \frac{1}{4} = \frac{\alpha - 1}{4} + \frac{r}{2} \qquad \text{with} \qquad s \geq r. \tag{7.110}$$

The residues of these poles are described by products of three point functions of on-shell states. These are the only poles that occur for $j_i > 0$.

Summarizing, the pole structure for $j_i > 0$ and the symmetries of the integral are fully captured, if we make the following ansatz for the amplitude

$$\widetilde{\mathcal{A}}(j_1, j_2, j_3, j_4) = f(j_1, j_3, s)\Delta(-4j_1 - 1)\Delta(-4j_3 - 1)\Delta\left(4\left(j_1 + j_3 - \frac{s}{2} + 1\right) - 1\right). \tag{7.111}$$

To determine the four-point amplitude we have to find the function $f(j_1, j_3, s)$.

High Energy Behavior

Next, we would like to understand the asymptotic behavior of the amplitudes for high energies. We consider, for example, the j_1 dependence. Introducing the notation $\alpha = -2\rho j_1$ we analyse the behavior of the integral in the limit $\alpha \to \infty$. Making the transformation of variables:

$$\eta = \exp\left(\frac{x}{\alpha}\right) \qquad \text{and} \qquad z_i = \exp\left(\frac{y_i}{\alpha}\right) \tag{7.112}$$

we are able to keep the relevant contributions to the integral (7.91). We obtain the asymptotic formula

$$\widetilde{\mathcal{A}}^{j_1\ j_2\ j_3\ j_4}_{m_1\ m_2\ m_3\ m_4} \approx \alpha^{8j_3 - 4s + 2} \qquad \text{for} \qquad \alpha \to \infty. \tag{7.113}$$

We can compute now the asymptotic behavior as $\alpha \to \infty$ of the r.h.s. of equation (7.111) using Stirling's formula. We obtain a power law growth as a function of α with the same exponent as in eq. (7.113). We conclude that in the asymptotic region $f(j_1, j_3, s)$ is independent of j_1 and, by the symmetry (7.93), of j_3 as well.

The results (7.111), (7.113) and the symmetries (7.93) and are actually enough to evaluate the whole amplitude in the case that the perturbation is a cosmological constant [34], because it is possible to show that the function $f(j_1, j_3, s)$ is an analytic function of j_1 and of j_3 as well. With the asymptotic behavior (7.113) and using powerful theorems for entire functions [75] one can conclude that f is independent of j_1 and j_3, so that $f = f(s)$. Then it was possible fix the unknown function $f(s)$ by choosing some convenient values of the momenta; by sending $N-3$ of the momenta to zero the amplitude is identified with derivatives of the three point function with respect to the cosmological constant and this fixes f uniquely. In the case of the black hole we will assume that f is an entire function as well. Since it depends only on s in the asymptotic region, we know that f does not depend on j_1 and by symmetry it does not depend on j_3 in the whole j_i plane. Summarizing, the four point function of tachyon operators has the form

$$\widetilde{\mathcal{A}}(j_1, j_2, j_3, j_4) = f(s)\Delta(-4j_1 - 1)\Delta(-4j_3 - 1)\Delta\left(4\left(j_1 + j_3 - \frac{s}{2} + 1\right) - 1\right). \tag{7.114}$$

The computation of $f(s)$ is more complicated in this case because the interaction is a discrete state and not a tachyon. However, we can calculate a four-point function where some of the $j_i's$ are fixed at the most convenient values, while keeping s arbitrary. Taking

$$j_1 = -\frac{1}{4} - \frac{\varepsilon}{4}, \tag{7.115}$$

where ε is a small parameter, we obtain from the ansatz (7.111) that the order $1/\varepsilon$ is given by the expression

$$\widetilde{\mathcal{A}}(j_1, j_2, j_3, j_4) \approx \frac{1}{\varepsilon}\Delta(-4j_2 - 1)\Delta(-4j_3 - 1)f(s). \tag{7.116}$$

On the other hand, we can directly compute the order $1/\varepsilon$ of this amplitude using the factorization formula (7.98). In this case, we observe that the residuum is described by a three-point function (7.107) of on-shell tachyons, that we already know how to calculate. Therefore, we have to consider

$$\widetilde{\mathcal{A}}^{j_1\ j_2\ j_3\ j_4}_{m_1\ m_2\ m_3\ m_4} \approx \frac{-\pi}{(4j_1+1)(4j_4+1)} \langle \mathcal{V}^{-}_{j_1+j_4\ m_1+m_4}(0) \mathcal{V}^{+}_{j_2\ m_2}(\infty) \mathcal{V}^{+}_{j_3\ m_3}(1) \rangle, \quad (7.117)$$

taking into account the kinematical constraints. After using our expression for the three-point function (7.62) we obtain the result

$$f(s) = -\pi(2s+1)\frac{\Delta(-4j_4-1)}{\Gamma(-s)}. \quad (7.118)$$

In this computation we have assumed that we have no other contributions to the residuum, which may come from the scaling of the screenings, while leaving η fixed. If this contribution is different from zero, the function $f(s)$ will be different. This point has to be carefully understood. Our result for the four-point function of tachyons in the black hole background is

$$\mathcal{A}(j_1, j_2, j_3, j_4) = \widetilde{M}^s(2s+1)\prod_{i=1}^{4}(-\pi)\Delta(-4j_i-1). \quad (7.119)$$

We have introduced a factor $(-\pi)^3$ into the definition of the path integral. This is in agreement with the standard notation of $c=1$. From the above result we see that we have agreement with the four-point function of tachyon operators in $c=1$ coupled to Liouville theory perturbed by the cosmological constant, if we take into account that the number of screenings between both theories differ by a factor of two and that the cosmological constant is square of the black hole mass. To do this computation we have used the three-point function of on-shell tachyons as a basic ingredient. We can conclude, that the regularisation procedure previously used for the three-point function is fully consistent with the four-point function.

8. CONCLUDING REMARKS

Quantum black hole physics involves many problems, which are hard to solve in four dimensions. One of them is the determination of the endpoint of black hole evaporation. The first difficulty to be faced is that at the late stages in the evaporation process quantum gravity effects become relevant. Therefore string theory could play an important role. A model to describe the propagation of strings in a black hole background was proposed by Witten [5]. The exact CFT describing a black hole in two dimensions has a Lagrangian formulation in terms of a gauged WZW model based on the non-compact group SL(2,$\mathbb{R}$). This model is exact in the sense that it solves the β-function equations of the string to all orders in the string coupling constant. This is important because, near the singularities, higher-order effects in α' are expected to be relevant. Since this black hole is described in terms of a coset model we can hope to address problems in black hole physics using well-known methods of CFT.

One of these well-known methods is the free-field description of a CFT. This formulation of Witten's black hole has been proposed by Bershadsky and Kutasov [27]. The problem of considering scattering processes becomes much simpler in this formulation. Using the Wakimoto representation of the SL(2,$\mathbb{R}$) current algebra, they obtained the action formulated in terms of Wakimoto coordinates. With this approach the SL(2,$\mathbb{R}$) symmetry of the theory is manifest, while a natural derivation of the action coming directly from the Lagrangian of the WZW model can be achieved using the Gauss decomposition [29]. In the semiclassical limit, the space-time structure of this model has been analysed in ref. [27]. It was found that it reproduces all the regions of an ordinary Schwarzschild black hole of classical general relativity.

However, Dijkgraaf, E. Verlinde and H. Verlinde [7] have shown that Witten's solution is only correct in the semiclassical limit of $k \to \infty$. The exact expressions for the dilaton and the metric receive corrections of order $1/k$ that can be computed with a mini-superspace description of the conformal field theory.

Carrying out the quantum mechanical analysis of the black hole in terms of Wakimoto coordinates, we have been able to find the same space-time interpretation for finite k [30]. This is important for two reasons. First it clarifies the relation between the free-field model of ref. [27] and Witten's black hole for finite k and shows the equivalence between the two models. Secondly it provides us with a space-time interpretation for $k = 9/4$ that is the interesting case in order to perform the computation of the scattering amplitudes of tachyons in the black hole background. It would be nice to derive the obtained expressions for the metric and the dilaton for finite k directly from the Lagrangian of ref. [27] with a careful treatment of the quantum effective action [8].

Having a suitable formulation of the exact 2D black hole solution of string theory we have performed the computation of the amplitudes describing the interaction of tachyons in the Euclidean black hole background. Specially interesting is the question whether there exists a connection between these scattering amplitudes and the $\mathcal{S}$-matrix describing the interaction of tachyons in standard non-critical string theory and the deformed matrix model of Jevicki and Yoneya [16]. These are systems for which we have a powerful nonperturbative description in terms of the matrix model formalism so that we can study the issues of singularities in string theory.

Using the Wakimoto free-field representation of the SL(2,$\mathbb{R}$)/U(1) Euclidean black hole, we have found that tachyon two-, three- and four-point correlation functions share a remarkable analogy with the tachyon amplitudes of $c = 1$ coupled to Liouville at non-vanishing cosmological constant. This observation was made by Bershadsky and Kutasov for those amplitudes where no screening charge is needed to satisfy the total charge balance.

In order to have non-vanishing correlation functions, we have infinitely renormalized the black hole mass. This is a well-known phenomenon for $c = 1$, and in this case it has interesting physical consequences. Perhaps this is the case for the black hole mass as well; further investigation is desirable. The amplitudes factorize in leg factors, which have poles at all the discrete states of $c = 1$. The new discrete states of Distler and Nelson [22] do not appear, because they are BRST-trivial in the Wakimoto representation, as checked in ref. [27] for the first examples. This is in agreement with the analysis of the BRST cohomology carried out by Eguchi et al. [24]. The scaling of the correlators is different from $c = 1$, but can be reproduced with the substitution $\mu^2 = M$. With our renormalization of the operators, we do not reproduce the pole structure of the correlators of the deformed matrix model [16]. Whether this discrepancy could be merely a normalization of the operators was discussed.

There are many interesting questions, suggested by these observations. An important one is to see whether this relation to $c = 1$ persists for the N-point functions. More results in this direction will appear in ref. [30]. It would be nice to see if the correlators can be computed using the Ward identities of $c = 1$ [76].

Previously there appeared several papers in the literature, where a relation between the 2D black hole and standard non-critical string theory is found [14,15,33]. It will be interesting to explore whether it is possible to see a direct connection to these approaches.

Recently Vafa and Mukhi [77] have proposed a topological field theory with which it is possible to compute tachyon correlation functions in non-critical string theory on higher-genus Riemann surfaces in the continuum approach [78,79]. The generalization of their methods to correlation functions of discrete states could be used to determine the $\mathcal{S}$-matrix of Witten's black hole for non-spherical topologies in the continuum formulation.

An important question is the connection between the considered 2D black hole solution and more realistic black holes in four dimensions [80,81,82]. In higher dimensions there exists an infinite number of propagating massive string modes and it would be interesting to analyse their role in black holes physics. Perhaps some new stringy effects will be discovered which will lead us to the conclusion that string theory is really the correct quantum theory of gravitation.

Acknowledgements

I would like to thank my advisor W. Nahm for giving me the opportunity to carry out the research work of my thesis at the Theory Division of CERN. I am grateful to my advisor L. Alvarez-Gaumé for his guidance and important discussions during these years. Very specially I would like to thank M. Becker for a fruitful collaboration. It is a pleasure to acknowledge discussions with L. Alvarez-Gaumé, J. L. Barbón, M. Bershadsky, U. Danielsson, J. Distler, V. Dotsenko, M. Douglas, E. Kiritsis, D. Kutasov, W. Lerche, G. Moore, W. Nahm, R. Plesser, N. Seiberg, S. Shenker, A. Tseytlin, E. Verlinde, H. Verlinde, C. Vafa and E. Witten. I would like to thank W. Nahm for a careful reading of the manuscript. Finally, I thank the Theory Division of CERN for hospitality and the Deutsche Forschungsgemeinschaft for financial support.

APPENDIX

For convenience we will collect the identities of Γ functions that we have used in our computations:

$$\Gamma(1+z-n) = (-1)^n \frac{\Gamma(1+z)\Gamma(-z)}{\Gamma(n-z)} \qquad \text{for} \qquad n \in \mathbb{N}, \tag{A.1}$$

$$\prod_{i=0}^{n-1}(i+x) = \frac{\Gamma(n+x)}{\Gamma(x)}, \quad \Delta(x)\Delta(-x) = -\frac{1}{x^2}, \quad \Delta(x)\Delta(1-x) = 1, \quad \Delta(1+x) = -x^2\Delta(x) \tag{A.2}$$

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\Big(x+\frac{1}{2}\Big), \qquad\qquad \Delta(2x) = 2^{4x-1}\Delta(x)\Delta\Big(x+\frac{1}{2}\Big). \tag{A.3}$$

To regularize the result of the singular integrals we use:

$$\lim_{\varepsilon\to 0}\Gamma(-n+\varepsilon) = \frac{(-)^n}{\varepsilon\Gamma(n+1)} + \mathcal{O}(1) \quad \text{for} \quad n \in \mathbb{N}. \tag{A.4}$$

The definition of the hypergeometric function, which we need in section 3 to compute the three-point function of generic tachyons, is:

$${}_3F_2(\alpha,\beta,\gamma;\rho,\sigma|x) = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\sum_{\nu=0}^{\infty}\frac{\Gamma(\alpha+\nu)\Gamma(\beta+\nu)\Gamma(\gamma+\nu)}{\Gamma(\rho+\nu)\Gamma(\sigma+\nu)}\frac{x^\nu}{\nu!}. \tag{A.5}$$

The following identity, known as “Dixon’s theorem”, is useful to evaluate the three-point function of on-shell tachyons:

$$ {}_3F_2(a,b,c;1{+}a{-}b,1{+}a{-}c|1) = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1-b-c+\frac{a}{2})}{\Gamma(1+a)\Gamma(1-b+\frac{a}{2})\Gamma(1-c+\frac{a}{2})\Gamma(1+a-b-c)}. \tag{A.6} $$

To evaluate the integrals for the three-point function, we have used the Dotsenko-Fateev (B.9) formula :

$$ \frac{1}{m!}\int \prod_{i=1}^{m}\Big(\frac{1}{2} i dz_i d\bar{z}_i\Big) \prod_{i=1}^{m} |z_i|^{2\alpha}|1-z_i|^{2\beta} \prod_{i<j}^{m} |z_i - z_j|^{4\rho} = \pi^m \big(\Delta(1-\rho)\big)^m $$

$$ \prod_{i=1}^{m} \Delta(i\rho) \prod_{i=0}^{m-1} \Delta(1+\alpha+i\rho)\Delta(1+\beta+i\rho)\Delta(-1-\alpha-\beta-(m-1+i)\rho). \tag{A.7} $$

REFERENCES

1. S. Hawking, "Particle Creation by Black Holes", *Commun. Math. Phys.* **43** (1975) 199.

2. J. A. Harvey and A. Strominger, "Quantum Aspects of Black Holes", Lectures given at the "*Trieste Spring School on String Theory and Quantum Gravity*", Trieste (1992) and at TASI 92: "*From Black Holes and Strings to Particles*", Boulder (1992), preprint EFI-92-41, hep-th/9209055; G. T. Horowitz, "The Dark Side of String Theory: Black Holes and Black Strings", Lectures given at the "*Trieste Spring School on String Theory and Quantum Gravity*", Trieste (1992), preprint UCSBTH-92-32, hep-th/9210119; S. Hawking, "Hawking on the Big Bang and Black Holes", Advanced Series in Astrophysics and Cosmology-Vol.8, World Scientific, 1993.

3. S. W. Hawking, "Breakdown of Predictability in Gravitational Collapse", *Phys. Rev.* **D14** (1976) 2460.

4. C. Callan, S. Giddings, J. Harvey and A. Strominger, "Evanescent Black Holes", *Phys. Rev.* **D45** (1992) 1005.

5. E. Witten, "String Theory and Black Holes", *Phys. Rev.* **D44** (1991) 314.

6. G. Mandal, A. Sengupta and S. Wadia, "Classical Solutions of Two-Dimensional String Theory", *Mod. Phys. Lett.* **A6** (1991) 1685.

7. R. Dijkgraaf, E. Verlinde and H. Verlinde, "String Propagation in Black Hole Geometry", *Nucl. Phys.* **B371** (1992) 269.

8. A. A. Tseytlin, "On the Form of the Black Hole Solution in $D = 2$ Theory", *Phys. Lett.* **B268** (1991) 175.

9. I. Jack, D. R. T. Jones and J. Panvel, "Exact Bosonic and Supersymmetric String Black Hole Solutions", *Nucl. Phys.* **B393** (1993) 95.

10. M. Perry and E. Teo, "Nonsingularity of the Exact Two-Dimensional String Black Hole", *Phys. Rev. Lett.* **70** (1993) 2669.

11. P. Yi, "Nonsingular 2D Black Holes and Classical String Backgrounds", *Phys. Rev.* **D48** (1993) 2777.

12. E. Witten, "Two-Dimensional String Theory and Black Holes", Lecture given at the "*Conference on Topics in Quantum Gravity*", Cincinnati (1992), preprint IASSNS-HEP-92-24, hep-th/9206069.

13. N. Seiberg and S. Shenker, "A Note on Background (In-)Dependence", *Phys. Rev.* **D45** (1992) 4581.

14. S. R. Das, "Matrix Models and Black Holes", *Mod. Phys. Lett.* **A8** (1993) 69; S. R. Das, "Matrix Models and Nonperturbative String Propagation in Two-dimensional Black Hole Backgrounds", *Mod. Phys. Lett.* **A8** (1993) 1331.

15. A. Dhar, G. Mandal and S. Wadia, "Wave Propagation in Stringy Black Hole", *Mod. Phys. Lett.* **A8** (1993) 1701; A. Dhar, G. Mandal and S. Wadia, "Stringy Quantum Effects in Two-dimensional Black Hole", *Mod. Phys. Lett.* **A7** (1992) 3703.

16. A. Jevicki and T. Yoneya, "A Deformed Matrix Model for the Black Hole Background in Two-Dimensional String Theory", *Nucl. Phys.* **B411** (1994) 64; K. Demeterfi and J. P. Rodrigues, "States and Quantum Effects in the Collective Field Theory of a Deformed Matrix Model", *Nucl. Phys.* **B415** (1994) 3; U. Danielsson, "A Matrix Model Black Hole", *Nucl. Phys.* **B395** (1993) 395; K. Demeterfi, I. Klebanov and J. P. Rodrigues, "The exact S Matrix of the Deformed $c = 1$ Matrix Model", *Phys. Rev. Lett.* **71** (1993) 3409; U. Danielsson, "The Deformed Matrix Model at Finite Radius and a New Duality Symmetry", preprint CERN-TH-7021-93, hep-th/9309157.

17. E. Brézin and V. A. Kazakov, "Exactly Solvable Field Theories of Closed Strings", *Phys. Lett.* **236B** (1990) 144; M. R. Douglas and S. H. Shenker,

"Strings in Less than One Dimension", *Nucl. Phys.* **B335** (1990) 635; D. J. Gross and A. A. Migdal, "Non-perturbative Two-Dimensional Quantum Gravity", *Phys. Rev. Lett.* **64** (1990) 127; D. J. Gross and A. A. Migdal, "A Non-perturbative Treatment of Two-Dimensional Quantum Gravity", *Nucl. Phys.* **B340** (1990) 333.

18. L. Alvarez-Gaumé, K. Becker and M. Becker, "Superloop Equations in the Double Scaling Limit", preprint CERN-TH.6575/92 (July 1992).
19. L. Alvarez-Gaumé, K. Becker, M. Becker, R. Emparán and J. Mañes, "Double Scaling Limit of the Super-Virasoro Constraints", *Int. J. Mod. Phys.* **A8** (1993) 2297.
20. L. Alvarez-Gaumé, K. Becker and M. Becker, "Super-Virasoro Constraints and Two-Dimensional Supergravity", talk given at the XIX International Colloquium on Group Theoretical Methods in Physics, Salamanca, Spain, 1992, preprint CERN-TH.6720/92.
21. K. Becker and M. Becker, "Non-perturbative Solution of the Super-Virasoro Constraints", *Mod. Phys. Lett.* **A8** (1993) 1205.
22. J. Distler and P. Nelson, "New Discrete States of Strings Near a Black Hole", *Nucl. Phys.* **B374** (1992) 123.
23. S. Chaudhuri and J. Lykken, "String Theory, Black Holes and SL(2,$\mathbb{R}$) Current Algebra", *Nucl. Phys.* **B396** (1993) 270.
24. T. Eguchi, H. Kanno and S. Yang, "W_∞ Algebra in Two-Dimensional Black Hole", *Phys. Lett.* **B298** (1993) 73.
25. J. Ellis, N. E. Mavromatos and D. V. Nanopoulos, "Quantum Coherence and Two-Dimensional Black Holes", *Phys. Lett.* **B267** (1991) 465.
26. N. Marcus and Y. Oz, "The Spectrum of 2D Black Hole or does the 2D Black Hole have Tachyonic W Hair?", *Nucl. Phys.* **B407** (1993) 429.
27. M. Bershadsky and D. Kutasov, "Comment on Gauged WZW Theory", *Phys. Lett.* **B266** (1991) 345.

28. A. Gerasimov, A. Morozov, M. Olshanetsky and A. Marshakov, "Wess-Zumino-Witten Model as a Theory of Free-Fields", *Int. J. Mod. Phys.* **A5** (1990) 2495.

29. C. Vafa and E. Verlinde, unpublished.

30. K. Becker and M. Becker, "More Comments on Gauged WZW Theory", preprint CERN-TH.7213/94, in preparation.

31. M. Goulian and M. Li, "Correlation Functions in Liouville Theory", *Phys. Rev. Lett.* **66** (1991) 2051.

32. K. Becker and M. Becker, "Interactions in the SL(2,$\mathbb{R}$)/U(1) Black Hole Background", *Nucl. Phys.* **B418** (1994) 206.

33. E. Martinec and S. L. Shatashvili, "Black Hole Physics and Liouville Theory", *Nucl. Phys.* **B368** (1992) 338.

34. P. Di Francesco and D. Kutasov, "World Sheet and Space Time Physics in Two Dimensional (Super-)String Theory", *Nucl. Phys.* **B375** (1992) 119; "Correlation Functions in 2D String Theory", *Phys. Lett.* **B261** (1991) 385.

35. See for example: C. W. Misner, K. S. Thorne and J. A. Wheeler, "Gravitation", Freeman, San Francisco (1973); or S. L. Shapiro and S. A. Teukolsky, "Black Holes, White Dwarfs and Neutron Stars", John Wiley & Sons, (1983).

36. See for example: M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory", Cambridge (1987).

37. E. Martinec, "An Introduction to 2-d gravity and Solvable String Models", Lectures given at the "*Trieste Spring School on String Theory and Quantum Gravity*", Trieste (1991), preprint RU-91-51, hep-th/9112019.

38. C. G. Callan, E. J. Martinec, M. J. Perry and D. Friedan, "Strings in Background Fields", *Nucl. Phys.* **B262** (1985) 593.

39. D. H. Friedan, "Nonlinear Models in $2+\epsilon$ Dimensions", *Ann. Phys.* **163** (1985) 318.

40. F. David, "Conformal Field Theories Coupled to 2-d Gravity in the Conformal Gauge", *Mod. Phys. Lett.* **A3** (1988) 1651.

41. J. Distler and H. Kawai, "Conformal Field Theory and 2D Quantum Gravity or Who's Afraid of Joseph Liouville?, *Nucl. Phys.* **B321** (1989) 509.

42. T. Banks, "The Tachyon Potential in String Theory", *Nucl. Phys.* **B361** (1991) 166.

43. E.Witten, "Non-Abelian Bosonization in Two Dimensions", *Commun. Math. Phys.* **92** (1984) 455.

44. A. M. Polyakov and P. B. Wiegmann, "Theory of Non-Abelian Goldstone Bosons", *Phys. Lett.* **131B** (1983) 121.

45. V. G. Knizhnik and A. B. Zamolodchikov, "Current Algebra and Wess-Zumino Model in Two Dimensions", *Nucl. Phys.* **B247** (1984) 83.

46. M. Kaku, "Strings, Conformal Fields, and Topology: an Introduction", Springer (1991).

47. K. Gawedzki and A. Kupiainen, "Coset Construction From Functional Integrals", *Nucl. Phys.* **B320** (1989) 625.

48. I. Bars, "String Propagation on Black Holes", preprint USC-91-HEP-B3, May 1991.

49. N. Ja. Vilenkin and A. U. Klimyk, "Representation of Lie Groups and Special Function", (Kluwer Academic Publishers, Dordrecht, 1992).

50. W. J. Holman and L. C. Biedenharn, "A General Study of the Wigner Coefficients of SU(1,1)", *Ann. Phys.* **47** (1968) 205; V. Bargman, "Irreducible Unitary Representations of the Lorentz Group", *Ann. Math.* **48** (1947) 568.

51. D. Karabali, Q-H. Park, H. Schnitzer, Z. Yang, "A GKO Construction Based on a Path Integral Formulation of Gauged Wess-Zumino-Witten Actions", *Phys. Lett.* **B216** (1989) 307; D. Karabali and H. Schnitzer,

"BRST Quantization of the Gauged W-Z-W Action and Coset Conformal Field Theories", *Nucl. Phys.* **B329** (1990) 649.

52. A. Giveon, "Target Space Duality and Stringy Black Holes", *Mod. Phys. Lett.* **A6** (1991) 2843.

53. M. Wakimoto, "Fock Representations of the Affine Lie Algebra ${A_1}^{(1)}$", *Commun. Math. Phys.* **104** (1986) 605.

54. V. S. Dotsenko, "The Free Field Representation of the SU(2) CFT", *Nucl. Phys.* **B338** (1990) 747; "Solving the SU(2) CFT with the Wakimoto Free Field Representation", *Nucl. Phys.* **B358** (1991) 547.

55. V. A. Fateev and A. B. Zamolodchikov, "Operator Algebra and Correlation Functions in the Two-Dimensional SU(2)$\times$ SU(2) Chiral Wess-Zumino Model", *Sov. J. Nucl. Phys.* **43** (4) (1986) 637.

56. D. Friedan, E. Martinec and S. Shenker, "Conformal Invariance, Supersymmetry and String Theory", *Nucl. Phys.* **B271** (1986) 93.

57. V. S. Dotsenko and V. A. Fateev, "Four-Point Correlation Function and the Operator Algebra in the 2D Conformal Invariant Theories with Central Charge $c < 1$", *Nucl. Phys.* **B251** (1985) 691; V. S. Dotsenko and V. A. Fateev, "Conformal Algebra and Multipoint Correlation Functions in 2D Statistical Models", *Nucl. Phys.* **B240** (1984) 312.

58. M. Goulian and M. Li, "Correlation Functions in Liouville Theory", *Phys. Rev. Lett.* **66** (1991) 2051.

59. V. S. Dotsenko, "Correlation Functions of Local Operators in 2D Gravity Coupled to Minimal Matter", Summer School Cargèse 1991, hep-th/9110030; "Three-Point Correlation Functions of the Minimal Conformal Field Theories Coupled to 2-D Gravity", *Mod. Phys. Lett.* **A6** (1991) 3601.

60. C. Vafa, "Strings and Singularities", preprint HUTP-93-A028, hep-th/9310069.

61. N. Seiberg, "Notes on Quantum Liouville Theory and Quantum Gravity", *Prog. Theor. Phys. Suppl.* **102** (1990) 319.

62. P. Ginsparg and G. Moore, "Lectures in 2D Gravity and 2D String Theory", preprint YCTP-P23-92, LA-UR-92-3479, hep-th/9304011.

63. D. J. Gross, I. R. Klebanov, M. J. Newman, "The Two-Point Correlation Function of the One-Dimensional Matrix Model", *Nucl. Phys.* **B350** (1991) 621; E. Witten, "Ground Ring of Two-Dimensional String Theory", *Nucl. Phys.* **B373** (1992) 187; A. M. Polyakov, "Selftuning Fields and Resonant Correlations in 2-d Gravity", *Mod. Phys. Lett.* **A6** (1991) 635.

64. A. A. Tseytlin, "Effective Action of the Gauged WZW Model and Exact String Solutions", *Nucl. Phys.* **B399** (1993) 601.

65. E. Witten and B. Zwiebach, "Algebraic Structures and Differential Geometry in 2D String Theory", *Nucl. Phys.* **B377** (1992) 55.

66. A. Gupta, S. Trivedi and M. Wise, "Random Surfaces in Conformal Gauge", *Nucl. Phys.* **B340** (1990) 475.

67. N. Sakai and Y. Tanii, "Operator Product Expansion and Topological States in $c = 1$ Matter Coupled to 2D Gravity", *Prog. Theor. Phys. Supp.* **110** (1992) 117.

68. M. B. Green and N. Seiberg, "Contact Interactions in Superstring Theory", *Nucl. Phys.* **B299** (1988) 559.

69. A. Erdélyi, "Higher Transcendental Functions", McGraw-Hill, New York, 1953, p. 188-189.

70. M. Li, "Correlators of Special States in $c = 1$ Liouville Theory", *Nucl. Phys.* **B382** (1992) 242.

71. J. L. F. Barbon, "Perturbing the Ground Ring of 2D String Theory", *Int. J. Mod. Phys.* **A7** (1992) 7579.

72. T. Eguchi, "$c = 1$ Liouville Theory Perturbed by the Black Hole Mass Operator", *Phys. Lett.* **B316** (1993) 74.

73. G. Moore, "Finite in All Directions", preprint YCTP-P12-93, hep-th/9305139.

74. E. Abdalla, M. C. B. Abdalla, D. Dalmazi and A. Zadra, "2D-Gravity in Non-Critical Strings: Continuum and Discrete Approaches", Lecture Notes in Physics, Series M, Springer Verlag (to appear).

75. R. Boas, "Entire Functions", Academic Press, NY 1954.

76. I. R. Klebanov, "Ward Identities in Two-Dimensional String Theory", *Mod. Phys. Lett.* **A7** (1992) 723; I. R. Klebanov and A. Pasquinucci, "Correlation Functions From Two-Dimensional String Ward Identities", *Nucl. Phys.* **B393** (1993) 261.

77. S. Mukhi and C. Vafa, "Two-Dimensional Black Hole as a Topological Coset of $c = 1$ String Theory", *Nucl. Phys.* **B407** (1993) 667.

78. D. Ghoshal and S. Mukhi, "Topological Landau-Ginzburg Model of Two-Dimensional String Theory", preprint MRI-PHY-13-93, hep-th/9312189; S. Mukhi, "The Two-Dimensional String as a Topological Field Theory, preprint TIFR-TH-93-61, hep-th/9312190.

79. A. Hanany, Y. Oz and R. Plesser, "Topological Landau Ginzburg Formulation and Integrable Structure of 2D String Theory", preprint IASSNS-HEP-94/1, hep-th/9401030

80. S. B. Giddings, J. Polchinski and A. Strominger, "Four Dimensional Black Holes in String Theory", *Phys. Rev.* **D48** (1993) 5784.

81. S. B. Giddings, J. A. Harvey, J. G. Polchinski, S. H. Shenker and A. Strominger, "Hairy Black Holes in String Theory", preprint NSF-ITP-93-58 hep-th/9309152.

82. G. T. Horowitz and D. L. Welch, "Exact Three Dimensional Black Holes in String Theory", *Phys. Rev. Lett.* **71** (1993) 328.